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On the morphism of Duflo–Kirillov type

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Abstract

We give the detailed proofs of some of Kontsevich's claims in the paper "Deformation Quantization of Poisson Manifolds I", i.e., we prove the compatibility of the two cup products, and prove two conjectures by using the formalism of the proof of Kontsevich's proof of his Formality theorem; the conjecture of Raïs, Kashiwara and Vergne and the conjecture of Bar-Natan, Garoufalidis, Rozansky and Thurston. Moreover, we calculate how the signatures appear.

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1. Introduction

Our purpose in this paper is to understand some of Kontsevich's claims in his excellent paper [10]. More precisely, we give the detailed proofs of the following:

1. The compatibility of the two cup products.
2. The conjecture of Raïs, Kashiwara and Vergne which we call the RKV conjecture.
3. The conjecture of Bar-Natan, Garoufalidis, Rozansky, and Thurston which we call the BGRT conjecture.

1.1. Formality theorem

In his celebrated paper [10], Kontsevich proved his Formality theorem which we explain first. From a manifold X, we obtain the differential graded Lie algebras $T_{\text{poly}}(X)$ and $D_{\text{poly}}(X)$ as follows:

$$T_{\text{poly}}(X)^i := \Gamma(X, \Lambda^{i+1}TX), \qquad D_{\text{poly}}(X)^i := \bigotimes_{C^\infty(X)}^{i+1} \text{Diff}(X).$$

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There is the natural quasi-isomorphism of complexes $T_{\mathrm{poly}}(X) \to D_{\mathrm{poly}}(X)$ by the following correspondence: for any $\xi_i \in \Gamma(X, TX)(i = 0, \dots, n)$

$$\begin{aligned}\mathcal{U}_1^{(0)} : T_{\mathrm{poly}}(X)^n &\ni \xi_0 \wedge \cdots \wedge \xi_n \\ &\mapsto \left(f_0 \otimes \cdots \otimes f_n \mapsto \sum_{\sigma \in \Sigma_{n+1}} \prod \mathrm{sign}(\sigma) \xi_{\sigma_i}(f_i) \right) \in D_{\mathrm{poly}}(X)^n.\end{aligned}$$

But it is not a morphism of differential graded Lie algebras.

Theorem 1.1 (Kontsevich). *There is the L_∞ quasi-isomorphism*

$$\mathcal{U} : C(T_{\mathrm{poly}}(X)) \to C(D_{\mathrm{poly}}(X)),$$

such that $\mathcal{U}_1 : T_{\mathrm{poly}}(X)[1] \to D_{\mathrm{poly}}(X)[1]$ *coincides with* $\mathcal{U}_1^{(0)}$.

Let $\alpha \in T_{\mathrm{poly}}(\mathbb{R}^d)$ be a solution of the Maurer–Cartan equation for $T_{\mathrm{poly}}(\mathbb{R}^d)$

$$d\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0.$$

We put $\tilde{\alpha} = \sum_{n \geq 1} (t^n/n!) \mathcal{U}_n(\overbrace{\alpha \wedge \cdots \wedge \alpha}^{n})$ which is a solution of the Maurer–Cartan equation for $D_{\mathrm{poly}}(\mathbb{R}^d)$. The α and $\tilde{\alpha}$ give the formal deformation of differential graded Lie algebra $T_{\mathrm{poly}}(\mathbb{R}^d)_\alpha[[t]]$ and $D_{\mathrm{poly}}(\mathbb{R}^d)_{\tilde{\alpha}}[[t]]$. The two complex $T_{\mathrm{poly}}(\mathbb{R}^d)_\alpha[-1]$ and $D_{\mathrm{poly}}(\mathbb{R}^d)_{\tilde{\alpha}}[-1]$ have the natural cup product. Kontsevich gave the outline of the proof of the compatibility of two cup products in the cohomology level. Our first purpose is to make it more detail.

1.2. Some applications

Formality theorem and the compatibility of the cup products are expected to have many applications in various domains of mathematics.

As one of them, Kontsevich constructed the algebra homomorphism

$$\mathrm{Center}(\mathfrak{U}(\mathfrak{g})) \to (\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}},$$

and showed that it coincides with the Duflo–Kirillov isomorphism, where $\mathfrak{g}$ is a finite dimensional Lie algebra and $\mathfrak{U}(\mathfrak{g})$ is the enveloping algebra. Moreover, he suggested that two conjectures about the Duflo–Kirillov type morphism, the RKV conjecture and the BGRT conjecture, can be resolved by his idea.

1.2.1. The RKV conjecture

Following the paper [9], we explain the RKV conjecture. Let G be a finite dimensional Lie group and $\mathfrak{g}$ be the finite dimensional Lie algebra associated to the G. We put as follows:

$$\begin{aligned}\mathcal{Z}(\mathfrak{g}) &= \mathrm{Center}(\mathfrak{U}(\mathfrak{g})) = \{\text{bi-invariant differential operators}\}, \\ I(\mathfrak{g}) &= \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} = \left\{ \begin{array}{l} \text{differential operator with constant coefficients,} \\ \text{invariant under the adjoint action.} \end{array} \right\}\end{aligned}$$

There is the Duflo–Kirillov isomorphism $\Phi : \mathcal{Z}(\mathfrak{g}) \to I(\mathfrak{g})$ of algebras. When $\mathfrak{g}$ is semisimple, it is the Harish–Chandra isomorphism.

There is the PBW isomorphism $\mathrm{Sym}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$, and its restriction to $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$, which we also call PBW isomorphism.

The composition of the PBW homomorphism and the Duflo–Kirillov morphism can be regarded as follows:

$$\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \ni V \mapsto V \circ j(x)^{1/2}|_{x=0} \in \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}},$$

where $j(x) = \det((1 - \mathrm{e}^{-a\,dx})/a\,dx; \mathfrak{g}) : \mathfrak{g} \to \mathbb{R}$. Under the Fourier transformation, it can also be regarded as follows: for any polynomial f on $\mathfrak{g}^*$

$$f \mapsto j\left(\frac{d}{dx}\right)^{1/2} f.$$

We can also regard $\mathfrak{U}(\mathfrak{g})$ and $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ as the set of distributions with supports contained in $\{0\}$ and $\{e\}$, respectively.

Since the function $j(x)$ is analytic on a neighborhood, the above map Φ can be naturally extended on the distributions on any open set of the G. Take an open set $U \ni e$ of G and the open set $U' \ni 0$ of $\mathfrak{g}$ such that $\exp : U' \to U$ is diffeomorphic. We have the natural isomorphism of the set of distributions on U and U',

$$\exp_* : \mathcal{D}'(U'; \mathfrak{g}) \xrightarrow{\sim} \mathcal{D}'(U, G).$$

The extended Duflo–Kirillov morphism $\Phi : \mathcal{D}'(U; G) \to \mathcal{D}'(U'; \mathfrak{g})$ is given by the following correspondence:

$$u \mapsto j(x)^{1/2} \cdot (\exp_*)^{-1} u.$$

Hence, we obtain the natural map $\mathcal{D}'_e(G) \to \mathcal{D}'_0(\mathfrak{g})$, where $\mathcal{D}'_e(G)$ (resp. $\mathcal{D}'_0(\mathfrak{g})$) means the set of germs of distributions at e (resp. 0).

Let K be a cone in $\mathfrak{g} \simeq T_eG \simeq T_0\mathfrak{g}$. For any element u of the $\mathcal{D}'_e(G)$ (resp. the $\mathcal{D}'_0(\mathfrak{g})$), the germ of support of u is defined, which induces the cone $C_e(\mathrm{supp}\, u)$ (resp. $C_0(\mathrm{supp}\, u)$) in the tangent space $T_eG = T_0\mathfrak{g} = \mathfrak{g}$. We put

$$\mathcal{D}'_e(K, G) = \{u \in \mathcal{D}'_e(G) | C_e(\mathrm{supp}\, u) \subset K\},$$

$$\mathcal{D}'_0(K, \mathfrak{g}) = \{u \in \mathcal{D}'_0(\mathfrak{g}) | C_0(\mathrm{supp}\, u) \subset K\}.$$

We have the natural morphism $\Phi : \mathcal{D}'_e(K; G) \to \mathcal{D}'_0(K; \mathfrak{g})$. There are the adjoint G-actions Ad of G on the G and $\mathfrak{g}$. If a subset K is an invariant under the adjoint action, then there are the G-actions on the $\mathcal{D}'_e(K, G)$ and the $\mathcal{D}'_0(K, \mathfrak{g})$. We put $\chi_0(g) = |\det(\mathrm{Ad}(g); \mathfrak{g})|$. Then we put as follows:

$$\mathcal{I}'_e(K, G) = \{u \in \mathcal{D}'_e(K, G) | u(ghg^{-1}) = \chi_0(g)^{-1} u(h)\},$$

$$\mathcal{I}'_0(K, \mathfrak{g}) = \{u \in \mathcal{D}'_0(K, \mathfrak{g}) | u(\mathrm{Ad}(g)x) = \chi_0(g)^{-1} u(x)\}.$$

There is the natural morphism $\Phi : \mathcal{I}'_e(K; G) \to \mathcal{I}'_0(K, \mathfrak{g})$.

For any invariant cones K_1, K_2 in $\mathfrak{g}$ such that $K_1 \cap (-K_2) = \{0\}$, the convolution product

$$*_G : \mathcal{I}'_e(K_1, G) \times \mathcal{I}'_e(K_2, G) \to \mathcal{I}'_e(K_1 + K_2; G),$$

$$*_{\mathfrak{g}} : \mathcal{I}'_0(K_1, \mathfrak{g}) \times \mathcal{I}'_0(K_2, \mathfrak{g}) \to \mathcal{I}'_0(K_1 + K_2; \mathfrak{g})$$

can be defined.

The problem is the following.

Problem 1.1. Does the morphism Φ preserve the product structure? In other words, does the following identity hold?

$$\Phi(u *_G v) = \Phi(u) *_{\mathfrak{g}} \Phi(v).$$

The affirmative answer is of importance for the harmonic analysis. In particular, the local solvability of bi-invariant differential operators follows (see the papers [9,12]). In the paper [9], Kashiwara and Vergne proposed the conjecture with respect to Campbell–Hausdorff formula, and give the affirmative answer under the assumption that the conjecture is true.

Kontsevich suggests that the problem is solvable affirmatively based on his theory (see [10, Section 8]). Our second purpose is to give the precise proof of this claim, i.e., we prove the following theorem.

Theorem 1.2. *Φ preserves the product structure.*

1.2.2. BGRT conjecture

We explain the conjecture of BGRT (see [4] for the terminology). They give the special element

$$\Omega = \exp_{m_{\mathcal{B}}}\left(\sum_{n=1}^{\infty} b_{2n}\omega_{2n}\right),$$

which we call Duflo–Kirillov element (see the paper [5]). They define the action of Chinese character Γ on the space of Chinese characters: $\hat{\Gamma} : \mathcal{B} \to \mathcal{B}$, and they propose the following conjecture to calculate the Kontsevich integral.

Theorem 1.3 (Wheeling conjecture [5]). *The morphism $\hat{\Omega} : (\mathcal{B}, m_{\mathcal{A}}) \to (\mathcal{B}, m_{\mathcal{B}})$ is an algebra homomorphism.*

Kontsevich suggests that his method is applicable also to resolve this conjecture. Our third purpose is to assure his claim precisely.

1.2.3. Sketch of Kontsevich's construction

We give the sketch of Kontsevich's construction of Duflo–Kirillov morphism. We regard the algebra $\mathrm{Sym}(\mathfrak{g})$ as the polynomial ring on $\mathfrak{g}^*$.

Let $\alpha \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ be the canonical tensor which gives the bracket of Lie algebra, i.e., $\alpha(X, Y) = [X, Y]$. We put

$$\tilde{\alpha} = \sum_{n \geq 1} \frac{1}{n!} \mathcal{U}_n(\overbrace{\alpha \wedge \cdots \wedge \alpha}^{n}).$$

Since the α is a solution of the Maurer–Cartan equation for the differential graded Lie algebra $T_{\text{poly}}(\mathfrak{g}^*)$, the $\tilde{\alpha}$ gives a solution of the Maurer–Cartan equation for the $D_{\text{poly}}(\mathfrak{g}^*)$. Hence it induces the deformation of the associative algebra $\text{Sym}(\mathfrak{g})$, which we denote by $(\text{Sym}(\mathfrak{g}), \star)$.

From the solutions of Maurer–Cartan equation α and $\tilde{\alpha}$, we obtain the deformed complexes $T_{\text{poly}}(\mathfrak{g}^*)_\alpha$ and $D_{\text{poly}}(\mathfrak{g}^*)_{\tilde{\alpha}}$.

We can see that the $\text{Sym}(\mathfrak{g})$ is the degree 0 part of the complex $T_{\text{poly}}(\mathbb{R}^d)[-1]$. The tangent map $T_\alpha \mathcal{U}$ preserve the product structures in the cohomology level. In particular, by applying the 0th cohomology part, we obtain the algebra isomorphism

$$T_\alpha \mathcal{U} : \text{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \text{Center}(\text{Sym}(\mathfrak{g}), \star).$$

We can check that $X \star Y - Y \star X = [X, Y]$ for any element $X, Y \in \mathfrak{g}$. Hence there is the algebra morphism $\mathfrak{U}(\mathfrak{g}) \to (\text{Sym}(\mathfrak{g}), \star)$ by the universality of the enveloping algebra. We can show that it is isomorphism. In particular, we obtain the algebra isomorphism I_{alg}

$$\text{Center}(\mathcal{U}(\mathfrak{g})) \to \text{Center}(\text{Sym}(\mathfrak{g}), \star).$$

There is the PBW isomorphism

$$\text{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \text{Center}(\mathfrak{U}(\mathfrak{g})).$$

Kontsevich found the form of the morphisms $T_\alpha \mathcal{U}$ and $I_{\text{alg}} \circ I_{\text{PBW}}$. Then he arrived at the conclusion that the morphism $T_\alpha \mathcal{U} \circ I_{\text{alg}}^{-1} \circ I_{\text{PBW}}$ coincides with the Kirillov–Duflo morphism.

1.2.4. Outline of proofs of the conjectures

We can solve the two conjectures by the following program:

1. Assume that we have two sets A and B with product structures and that there is the 'PBW isomorphism' $I_{\text{PBW}} : B \to A$.
2. Using the Formality theorem or the combinatorics of it, we deform the product structure on the B canonically to construct $(B, \star)$. There is the canonical morphism $I_T B \to (B, \star)$. The compatibility of the cup products assures that the restriction of the map I_T to the appropriate subset of the B preserves the product structure.
3. We construct the isomorphism $I_{\text{alg}} : A \to B$ which satisfies the two conditions:
   - 3.1. it preserves the product structure;
   - 3.2. the combinatorial structure of the construction coincides with that of $\mathfrak{U}(\mathfrak{g}) \to (\text{Sym}(\mathfrak{g}), \star)$.
4. Then the composition of algebra isomorphisms $I_{\text{PBW}} \circ I_{\text{alg}^{-1}} \circ I_T$ coincides the Kirillov–Duflo type morphism, because the combinatorics of the construction is same as that of Kontsevich.

1.3. Contents of the paper

We explain the contents of this paper. First, we introduce a variation of configuration spaces $C_{n,m}, C_n$ and describe the stratification of the compactification. We define l-admissible graph and its weight, and construct $\mathcal{U}_n^l$ as a generalization of $\mathcal{U}_n$, i.e., we put $\mathcal{U}_n^l = \sum_\Gamma W_\Gamma \cdot \mathcal{U}_\Gamma$.

Following Kontsevich, we rewrite the Stokes formula, which leads the compatibility of the two cup products (see Section 3.4 for more precise statement).

In [10], the signatures are not written clearly. We tried to make it clear how and why the signatures appear, but the author is not certain whether his understanding of the formalism with respect to the signature is standard, for it seems that there are several signature rules. The rule we use in this paper is explained in Section 2.

In Sections 4 and 5, we give proofs of the RKV conjecture and the BGRT conjecture, respectively.

1.4. Related works

We should mention the very closely related works by others. The RKV conjecture has been proved by Andler et al. [1,2], and the BGRT conjecture has been proved by Thurston [13] by other methods. As for the signature, Arnal, Manchon and Masmoudi handled the problem (see [3]). The author believes that this paper is of value at least in our understanding the marvelous work of Kontsevich.

2. Preliminaries

2.1. The signature

We often identify two groups $\{1,-1\}$ and $\mathbb{Z}/2\mathbb{Z}$. For an element $\delta \in \{1,-1\}$, we denote the corresponding element of $\mathbb{Z}/2\mathbb{Z}$ by δ', i.e., $\{1,-1\} \ni \delta \leftrightarrow \delta' \in \mathbb{Z}/2\mathbb{Z}$.

In the following, we consider the vector spaces over the field $\mathbb{R}$ of real numbers. The graded vector space $\mathbf{1}[k]$ with generator $1[k]$ is defined to be as follows:

$$\mathbf{1}[k]^i = \begin{cases} \mathbb{R}, & i = -k, \\ 0, & i \neq -k. \end{cases}$$

For any graded vector space X, we regard the shift $X[k]$ as the $\mathbf{1}[k] \otimes X$. For any element x of the X, we denote $1[k] \otimes x$ by $x[k]$.

The dg-rule says that if we exchange the order of two objects with the degrees k and l, respectively, it appears the signature $(-1)^{kl}$. For example, we have the natural morphism

$$X[k_1] \otimes Y[k_2] \to (X \otimes Y)[k_1 + k_2],$$

$$x[k_1] \otimes y[k_2] = (1[k_1] \otimes x) \otimes (1[k_2] \otimes y) \mapsto (-1)^{\deg(x)k_2} 1[k_1 + k_2] \otimes (x \otimes y).$$

Based on the dg-rule, we follow the dg-composition rule in the paper, which we explain by the simple example. Let $X_i (i = 1, 2, 3, 4)$ be graded vector spaces. Assume that the operations $\circ : X_1 \otimes X_3 \to V$ and $\bullet : X_2 \otimes X_4 \to W$ are given. In this case the operation $\circ \otimes \bullet : \otimes X_i[k_i] \to V \otimes W[\sum k_i]$ is defined as follows: we regard the vector space $X[k]$ as Let x_i be any object of X_i with degree $|x_i|$:

1. Start from $x_1[k_1] \otimes x_2[k_2] \otimes x_3[k_3] \otimes x_4[k_4]$.
2. Bringing the shifts $1[k_i]\otimes$ to top, we obtain $x_1 \otimes \cdots \otimes x_4[\sum k_i]$. The signatures appear when we exchange the order of the $1[k_i]$ and x_j for $j < i$. Thus the signature $\sum |x_j| \cdot (\sum_{p=j+1}^{4} k_p)$ appears.
3. Exchanging the order of x_2 and x_3, we obtain $x_1 \otimes x_3 \otimes x_2 \otimes x_4[\sum k_i]$. The signature $|x_2| \cdot |x_3|$ appears.
4. We arrive at the element $(x_1 \circ x_3) \otimes (x_2 \bullet x_4)[\sum k_i]$.

Hence the total signature is $\sum |x_j|(\sum_{p=j+1}^{4} k_p) + |x_2| \cdot |x_3|$, i.e. $(\circ \otimes \bullet)(x_1[k_1] \otimes x_2[k_2] \otimes x_3[k_3] \otimes x_4[k_4])$ is defined to be $(-1)^{\sum |x_j|(\sum_{p=j+1}^{4} k_p)+|x_2|\cdot|x_3|}(x_1 \circ x_3) \otimes (x_2 \bullet x_4)[\sum k_i]$.

Let $V^{\cdot}$ be a graded vector space. For any n elements $\gamma_i \in V^{l_i}$ and any element $\sigma \in \Sigma_n$, the symmetric differential graded signature $\mathrm{sgn}(\sigma, (\gamma_p))$ is defined by the following formula:

$$\gamma_1 \cdots \gamma_n = \mathrm{sgn}(\sigma, (\gamma_p))\gamma_{\sigma_1} \cdots \gamma_{\sigma_n},$$

where $\cdot$ denotes the symmetric product in the graded vector spaces. On the other hand, the anti-symmetric differential graded signature $\mathrm{asgn}(\sigma, (\gamma_p))$ is defined by the following formula:

$$\gamma_1 \wedge \cdots \wedge \gamma_n = \mathrm{asgn}(\sigma, (\gamma_p))\gamma_{\sigma_1} \wedge \cdots \wedge \gamma_{\sigma_n}.$$

Remark 2.1. For any element $\sigma \in \Sigma_n$, we denote the usual signature of σ by $\mathrm{asgn}(\sigma)$.

For integers $l_p (p = 1, \dots, n)$, $\mathrm{sgn}(\sigma, (l_p))$ and $\mathrm{asgn}(\sigma, (l_p))$ are defined naturally.

If the grade $|\gamma_p|$ is even for each γ_p, the signature $\mathrm{sgn}(\sigma, (\gamma_p))$ is 1. On the other hand, if the grade $|\gamma_p|$ is odd for any p, then the signature $\mathrm{sgn}(\sigma, (\gamma_p))$ equals $\mathrm{asgn}(\sigma)$.

For any $i, j \in \{1, \dots, n\} (i < j)$, we denote by κ_{ij} the following permutation:

$$(1, \dots, n) \to (1, \dots, i-1, i, j, i+1, \dots, j-1, j+1, \dots, n),$$

and denote by κ_{ji} the following permutation:

$$(1, \dots, n) \to (1, \dots, i-1, j, i, i+1, \dots, j-1, j+1, \dots, n).$$

2.2. Two differential graded Lie algebras

Let A be a graded commutative algebra over the field $\mathbb{R}$ of real numbers, and $\mathfrak{g}$ be a differential graded Lie subalgebra of $\mathrm{Der}_{\mathbb{R}}(A)$. We assume that there are homogeneous elements $v_1, \dots, v_p \in \mathfrak{g}^{\cdot}$ such that $\mathfrak{g}^{\cdot} = \oplus_i A v_i$ and that $[v_i, v_j] = 0$. We have the action of $\mathfrak{g}$ on $\mathfrak{g}$ as $\gamma(\sum a_i v_i) = \sum \gamma(a_i) v_i$.

Example. We have the following examples:

$A = C^\infty(\mathbb{R}^{d|e})$: the ring of functions on super space $\mathbb{R}^{d|e}$. We put $\mathfrak{g} = \mathrm{Der}_{\mathbb{R}}(A)$.
$A = C^\infty(\mathbb{R}^d) \otimes R^\cdot$, where $R^\cdot$ graded Artinnian ring over $\mathbb{R}$. We put $\mathfrak{g} := \mathrm{Der}_R(A)$.

We can take the canonical basis v_i for $\mathfrak{g}$. We put as follows:

$$T_{\mathrm{poly}}(A^\cdot) := \bigoplus_{k\geq 0} \Lambda^k \mathfrak{g}^\cdot[-k+1], \qquad D^\cdot_{\mathrm{poly}}(A^\cdot) := \bigoplus_{k\geq 0} \bigotimes_A^k \mathrm{Diff}(A)[-k+1].$$

2.2.1. $T_{\mathrm{poly}}(A^\cdot)$

We give the differential graded Lie algebra structure to $T_{\mathrm{poly}}(A^\cdot)$ as follows: consider any two elements

$$\boldsymbol{\xi} = \xi_0 \wedge \cdots \wedge \xi_k \in \Lambda^{k+1}\mathfrak{g}^\cdot, \qquad \boldsymbol{\eta} = \eta_0 \wedge \cdots \wedge \eta_l \in \Lambda^{l+1}\mathfrak{g}^\cdot,$$

where ξ_i and η_j are homogeneous elements of $\mathfrak{g}^\cdot$. For $\boldsymbol{\xi}$, we put as follows:

$$\bar{\boldsymbol{\xi}}_i = \xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k.$$

We define the bracket operator as follows:

$$[\boldsymbol{\xi}[k], \boldsymbol{\eta}[l]] = \sum_{i=0}\sum_{j=0} \epsilon(i,j)([\xi_i, \eta_j] \wedge \bar{\boldsymbol{\xi}}_i \wedge \bar{\boldsymbol{\eta}}_j)[k+l], \tag{1}$$

where we put as follows:

$$\epsilon(i,j)' := i + j + |\xi_i| \left(\sum^{i-1} |\xi_p|\right) + |\eta_j| \left(\sum^{j-1} |\eta_q|\right) + |\eta_j| \left(\sum |\xi_p| - |\xi_i|\right) + l\left(\sum |\xi_p|\right) \pmod 2.$$

The terms $i + |\xi_i|(\sum^{i-1}|\xi_p|)$ and $j + |\eta_j|(\sum^{j-1}|\eta_q|)$ is due to the anti-commutativity of the wedge product. The term $|\eta_j|(\sum|\xi_p| - |\xi_i|) + l(\sum|\xi_p|)$ appears because of the dg-composition rule for the operation

$$(\mathfrak{g} \otimes \Lambda^k \mathfrak{g})[-k] \otimes (\mathfrak{g} \otimes \Lambda^l \mathfrak{g})[-l] \to (\mathfrak{g} \otimes \Lambda^{k+l}\mathfrak{g})[-k-l].$$

We define the operation $\bullet$ as follows:

$$\boldsymbol{\xi}[k] \bullet \boldsymbol{\eta}[l] = \sum_{i=0} (-1)^{k-i+|\xi_i|(\sum_{p>i}|\xi_p|)+(\sum|\xi_p|)l} \bar{\boldsymbol{\xi}}_i \wedge \xi_i(\boldsymbol{\eta})[k+l]. \tag{2}$$

We have the following equality in $\mathbb{Z}/2\mathbb{Z}$:

$$k - i + |\xi_i| \sum_{p>i} |\xi_p| + l \sum |\xi_p| + |\eta_j| \sum^{j-1} |\eta_q| + j + (|\xi_i| + |\eta_j|)\left(\sum |\xi_p| - |\xi_i|\right) + k \equiv \epsilon(i,j)'. \tag{3}$$

Thus it holds that

$$\boldsymbol{\xi}[k] \bullet \boldsymbol{\eta}[l] = \sum_{i=0}\sum_{j=0} \epsilon(i,j)(\xi_i(\eta_j) \wedge \bar{\boldsymbol{\xi}}_i \wedge \bar{\boldsymbol{\eta}}_j)[k+l]. \tag{4}$$

Hence, we obtain the following lemma.

Lemma 2.1. *The following equality holds*:

$$[\boldsymbol{\xi}[k], \boldsymbol{\eta}[l]] = \boldsymbol{\xi}[k] \bullet \boldsymbol{\eta}[l] - (-1)^{|\boldsymbol{\xi}[k]|\cdot|\boldsymbol{\eta}[l]|} \boldsymbol{\eta}[k] \bullet \boldsymbol{\xi}[l].$$

2.2.2. $D_{\text{poly}}(A)$

For a graded algebra $A^{\cdot}$ over a field k, we put $C_{\text{poly}}(A^{\cdot}) := \oplus_l \text{Hom}(A^{\cdot \otimes l}, A^{\cdot})[-l+1]$. The following operations on the C_{poly} are closed for the D_{poly}.

The grading of the vector space $\text{Hom}(A^{\cdot \otimes l}, A^{\cdot})$ is defined as follows: the degree of an element $\Psi \in \text{Hom}(A^{\cdot \otimes l}, A^{\cdot})$ is r if $\Psi = 0$ or if it holds that $|\Psi(x_1 \otimes \cdots \otimes x_l)| - \sum |x_i| = r$ for any homogeneous elements of $A^{\cdot}$. For the element $\Phi = \Psi[-l+1] \in \text{Hom}(A^{\cdot \otimes l}, A^{\cdot})[-l+1]$, we put $|\Phi|_1 := |\Psi|$ and $|\Phi|_2 := l-1$. We put that $|\Phi| = |\Phi|_1 + |\Phi|_2$.

For any $\Phi \in \text{Hom}(A^{\otimes r}, A)[-r+1]$ with $|\Phi|_2 = s$, which we call homogeneous, and for any homogeneous element $a_p \in A[1]$, we put as follows:

$$\Phi(a_1 \otimes \cdots \otimes a_r) := (-1)^{\sum (r-i+1)|a_i| + r\cdot|\Phi|} \Phi[r-1](a_1[-1] \otimes \cdots \otimes a_r[-1])[1],$$

where the signature is determined by the dg-composition rule. For any homogeneous elements $\Phi \in C_{\text{poly}}$ and $a_i \in A[1]$, we put as follows:

$$F_l^i(\Phi)(a_1 \otimes \cdots \otimes a_l) := (-1)^{\sum_p^{i-1}(|a_i|)|\Phi|} a_1 \otimes \cdots \otimes \Phi(a_i \otimes \cdots) \otimes \cdots \otimes a_l, \tag{5}$$

where the signature is obtained as the dg-symmetric signature of the transformation

$$\Phi \otimes a_1 \otimes \cdots \otimes a_l \mapsto a_1 \otimes \cdots \otimes \Phi \otimes a_i \otimes \cdots \otimes a_l.$$

We put $F_l(\Phi) := \sum_{i \geq 1} F_l^i(\Phi)$. Hence, we obtain the homomorphisms $F = (F_l)$ and $F^i = (F_l^i)$ from the $\text{Hom}(A^{\otimes r}, A)[-r+1]$ to the $\prod_m \text{Hom}(A^{\otimes m}, A^{\otimes m-r+1})[-r+1]$. We denote by π the natural projection

$$\prod_m \text{Hom}(A^{\otimes m}, A^{\otimes m-r+1})[-r+1] \to \text{Hom}(A^{\otimes r}, A)[-r+1].$$

We have that $\pi \circ F = \text{id}$. We define the composition $\Phi_1 \circ \Phi_2$ as $\Phi_1 \circ \Phi_2 := \pi(F(\Phi_1) \circ F(\Phi_2))$, where the second $\circ$ denotes the composition of the morphisms. Also, we define that $\Phi_1 \circ_i \Phi_2 = \pi(F(\Phi_1) \circ F^i(\Phi_2))$.

Usually, the differential graded Lie algebra structure of the C_{poly} is given by the Gerstenhaber bracket, i.e., for two homogeneous elements Φ_1 and Φ_2, the Gerstenhaber bracket $[,]'$ is defined as follows:

$$[\Phi_1, \Phi_2]' := \Phi_1 \circ \Phi_2 - (-1)^{|\Phi_1|\cdot|\Phi_2|} \Phi_2 \circ \Phi_1. \tag{6}$$

The differential d is defined as $d(x) = [m, x]'$, where m denotes the multiplication $A \otimes A \to A$. In this case, the Maurer–Cartan solution induces a formal A_∞-deformation of the associative structure m of the A, i.e., the formal deformation $\tilde{d}$ of the differential of $C_{\mathrm{poly}}(A)$. Also, it induces an A_∞-deformation of the associative differential graded algebra $C_{\mathrm{poly}}(A)[-1]$ as follows: we put $\bar{\mu} := \mu + m$. We have the decomposition $\bar{\mu} = \sum \mu_l$, where the μ_l belongs to the $\mathrm{Hom}(A^{\otimes l}, A^{\cdot})$. Consider elements $t_i \in \mathrm{Hom}(A^{\otimes r_i}, A)[-r_i+1]$ with $|t_i|_1 = s_i$, which we call homogeneous. We put for homogeneous elements a_p of the $A[1]$

$$\begin{aligned}&\square(t_1 \otimes \cdots \otimes t_n)(a_1 \otimes \cdots \otimes a_m)\\ &\quad= \sum \epsilon(\{t_i\}, \{a_p\}, \{k_q\}) a_1 \otimes \cdots t_i(a_{k_i} \otimes \cdots \otimes a_{k_i+r_i-1}) \otimes \cdots ,\end{aligned}$$

where we put $\epsilon(\{t_i\}, \{a_p\}, \{k_q\})' = (-1)^{\sum_{i=1}^{l}(|t_i|)\sum_{p}^{i-1}(|a_p|)}$ which is the dg-symmetric signature of the transformation $t_1 \otimes \cdots \otimes t_l \otimes a_1 \otimes \cdots \otimes a_m \mapsto a_1 \otimes \cdots \otimes (t_i \otimes a_i \otimes \cdots) \otimes \cdots$. Thus, we obtain the homomorphism

$$\square : \bigotimes_{i=1}^{n} \mathrm{Hom}(A^{\otimes r_i}, A) \to \prod_m \mathrm{Hom}\left(A^{\otimes m}, A^{\otimes (m - \sum (r_i - 1))}\right).$$

Define $\tilde{\mu}_l : \otimes^l C_{\mathrm{poly}} \to C_{\mathrm{poly}}$ for $l \geq 2$ as

$$\tilde{\mu}_l(t_1 \otimes \cdots \otimes t_l) := \pi(F(\bar{\mu}) \circ (\square(t_1 \otimes \cdots \otimes t_l))).$$

We put $\tilde{\mu}_1 = \mathrm{ad}(\bar{\mu})$.

We obtain the following lemma, which may be well known, by a direct calculation.

Lemma 2.2. *It holds that* $\tilde{\mu} \circ \tilde{\mu} = 0$.

Proof. We have the following equality:

$$\begin{aligned}&\tilde{\mu} \circ \tilde{\mu}(t_1 \otimes \cdots \otimes t_l)(a_1 \otimes \cdots \otimes a_m)\\ &\quad= \mu \circ \mu(\square(t_1 \otimes \cdots \otimes t_l)(a_1 \otimes \cdots \otimes a_m))\\ &\qquad -\tilde{\mu}\left(\sum (-1)^{\sum^{i-1} |t_p|} t_1 \otimes \cdots \otimes (t_i \circ \mu) \otimes \cdots \otimes t_l\right)(a_1 \otimes \cdots \otimes a_m)\\ &\qquad +\tilde{\mu}\left(\sum (-1)^{\sum^{i-1} |t_p|} t_1 \otimes \cdots \otimes (t_i \circ \mu) \otimes \cdots \otimes t_l\right)(a_1 \otimes \cdots \otimes a_m).\end{aligned} \tag{7}$$

Hence we are done. □

We change the bracket of $C_{\mathrm{poly}}(A)$ slightly as follows. For homogeneous elements $\Phi_i \in C_{\mathrm{poly}}$ the changed bracket [,] is defined as follows:

$$[\Phi_1, \Phi_2] := (-1)^{|\Phi_1|_2 \cdot |\Phi_2|_2} [\Phi_1, \Phi_2]'. \tag{8}$$

We put $\Phi_1 \bar{\circ} \Phi_2 = (-1)^{|\Phi_1|_2 |\Phi_2|_2} \Phi_1 \circ \Phi_2$.

From the Maurer–Cartan solution for this differential graded Lie algebra structure, we obtain the following structure which we call A'_∞-structure. We have the element $\bar{\mu} = \sum \mu_l$

as above. Then, we define the morphism $\tilde{\mu}(l \geq 1)$ as follows:

$$\begin{aligned}
&\tilde{\mu}_l(t_1 \otimes \cdots \otimes t_l)(f_1 \otimes \cdots \otimes f_m) \\
&\quad = \delta \cdot \pi(F(\mu) \circ (\Box(t_1 \otimes \cdots \otimes t_l)))(f_1 \otimes \cdots \otimes f_m) \quad (l \geq 2), \\
&\tilde{\mu}_1 = \mathrm{ad}(\mu) \quad (l = 1),
\end{aligned}$$

where we put as follows:

$$\delta' \equiv \sum_{i=1}^{l} |t_i|_2 \left(\sum_{j=i}^{l} |t_j|_2 + m - 1 \right) + l \pmod 2.$$

The following lemma can be shown by a direct calculation.

Lemma 2.3. *We have that* $\tilde{\mu} \circ \tilde{\mu} = 0$.

Proof. We obtain the following equality by direct calculation:

$$\begin{aligned}
&\tilde{\mu} \circ \tilde{\mu}(t_1 \otimes \cdots \otimes t_l)(a_1 \otimes \cdots \otimes a_m) \\
&\quad = \rho \sum_{a+b=m-\sum |t_i|_2} (-1)^{ab} \mu_a \circ \mu_b(\Box(t_1 \otimes \cdots \otimes t_l)(a_1 \otimes \cdots \otimes a_m)) \\
&\qquad + \sum_{a+b=m-\sum |t_i|_2} \mu_b \left(\sum (-1)^{\sum^{i-1} |t_p| + a|t_i|_2} t_1 \otimes \cdots \right. \\
&\qquad \left. \otimes t_i \circ \mu_a \otimes \cdots \otimes t_l \right) (a_1 \otimes \cdots \otimes a_m) \\
&\qquad - \sum_{a+b=m-\sum |t_i|_2} \mu_b \left(\sum (-1)^{\sum^{i-1} |t_p| + a|t_i|_2} t_1 \otimes \cdots \right. \\
&\qquad \left. \otimes t_i \circ \mu_a \otimes \cdots \otimes t_l \right) (a_1 \otimes \cdots \otimes a_m),
\end{aligned} \tag{9}$$

where we put

$$\rho' \equiv \sum |t_p|_2 \sum^{p-1} |t_q|_2 + \left(m - \sum |t_p|_2 \right) \sum_{p=1}^{l} |t_p| + l \pmod 2.$$

Hence we are done. □

2.3. Configuration spaces and its variation

Kontsevich introduced the configuration spaces $C_{n,m}$ and C_n to prove the Formality theorem. As a variation, we consider the following spaces $X^l_{n,m}$ and X^l_n: note the natural isomorphism $\varphi : C^{\mathbb{R}}_{l+1,0} \simeq \mathring{\Delta}^{l-1}$, where $\mathring{\Delta}^{l-1}$ denotes the interior of the standard $(l-1)$-simplex. We represent the map φ as $\varphi(x) = (p_0(x) = 0, \ldots, p_l(x) = 1)$. We obtain the map $\psi : \mathring{\Delta}^l \simeq \mathring{\Delta}^{l-1} \times \mathbb{R}_{>0} \to C_{l+1,0}$ as

$$\psi(x,t) = (p_i(x) + \sqrt{-1}t).$$

We put $\mathring{X}^l_{n,m} = C_{n,m} \times_{C_{l+1,0}} (\mathring{\Delta}^l)$. It has the natural orientation. We denote the closure of the $\mathring{X}^l_{n,m}$ in the $\bar{C}_{n,m}$ by $X^l_{n,m}$. As is easily seen, we have the equality $\dim X^l_{n,m} = 2n+m-2-l$. Similarly, we put $\mathring{X}^l_n = C_n \times_{C_{l+1}} \mathring{\Delta}^{l-1}$, where $\mathring{\Delta}^{l-1} \simeq C^{\mathbb{R}}_{l+1} \subset C_{l+1}$ and denote its closure in the $\bar{C}_n$ by X^l_n. We have that $\dim X^l_n = 2n - 3 - l$. We denote the closure of $\mathring{\Delta}^{l-1}$ in the $\bar{C}_{l+1}$ by D_l. We have $X^l_{n,m} = \bar{C}_{n,m} \times_{\bar{C}_{l+1,0}} D_l$.

The configuration spaces $\bar{C}_{n,m}$ have the natural stratifications defined by Kontsevich. They induce those on the D_l and the $\bar{X}^l_{n,m}$ by restriction. We describe them in the following. Observe that the $X^l_{n,m}$ and the D_l are locally polyhedron for an appropriate coordinate.

2.3.1. The stratification of $\bar{C}_{n,m}$

Consider the following data (T, α, ϕ, β) with the following conditions:

- T is an oriented tree with the unique root vertex and the unique root edge. We have the natural order on the set V_T of the vertices of T (take the root point to be the minimal point), and the natural order on the set E_T of the edges of the T (take the edge starting at the root point to be the minimal point).
- ϕ is the map $E_T \to \{1, 2\}$ such that it holds that $\phi(e) \geq \phi(e')$ if $e \leq e'$. We denote by $\mathrm{St}(u)$, for any vertex u, the set of edges e starting from u. We put $\mathrm{St}_i(u) = \{e \in \mathrm{St}(u) | \phi(e) = i\}$.

  Since T is an oriented rooted tree, any vertex u except the root has the unique incoming edge $e(u)$. Thus ϕ gives the function $\phi : V_T \to \{1, 2\}$ by the correspondence $\phi(u) = \phi(e(u)), \phi(\mathrm{root}) = 2$.
- α is the decomposition $\mathrm{En}(T) = \mathrm{En}_1(T) \sqcup \mathrm{En}_2(T)$ and the ordering of each set, where $\mathrm{En}(T)$ denotes the set of maximal edges. α_i denotes the numbering of $\mathrm{En}_i(T)$.
- $\beta = (\beta(u))$, where $\beta(u)$ is an ordering of $\mathrm{St}_2(u)$. Note that β gives the lexicographic order to $\mathrm{En}(T)$.
- If $|\mathrm{St}(u)| = 1$, then it holds that $\mathrm{St}(u) = \mathrm{St}_1(u)$, and $\phi(e) = 2$ for the edge ending at u.
- We assume that two orders on $\mathrm{En}_2(T)$ given by α and β coincide.

Definition 2.1. We call such data a numbered oriented 2-tree of type (n, m).

Let (T, α, ϕ, β) be a numbered oriented 2-tree, and T' be an oriented tree obtained by collapsing an edge $e : u \to v$ of T which is not maximal. We denote by $\bar{u}$, the vertex obtained by collapsing the edge e. Then the data $(T', \alpha', \phi', \beta')$ is induced from the (T, α, ϕ, β) as follows: we have the natural isomorphism $\mathrm{En}(T) \simeq \mathrm{En}(T')$ hence α induces α'. Since, we have the inclusion $E_{T'} \subset E_T$, the map ϕ' is obtained as the restriction. As is easily seen, ϕ' satisfies the condition. If $\phi(e) = 1$, then $\mathrm{St}_2(\bar{u}) = \mathrm{St}_2(u)$. Hence $\beta(u)$ induces $\beta'(\bar{u})$. When $\phi(e) = 2$, we have that $\mathrm{St}_2(\bar{u}) = (\mathrm{St}_2(u) - \{e\}) \sqcup \mathrm{St}_2(v)$. The orders $\beta(u), \beta(v)$ and the condition "for edges $e'(\neq e) \in \mathrm{St}_2(u), e'' \in \mathrm{St}_2(v), e' \leq e''$ if and only if $e' \leq e$" determine the unique order $\beta'(\bar{u})$ on the $\mathrm{St}_2(\bar{u})$. Hence the collection β of orders induces β'.

Similarly, if the T' is obtained by collapsing some edges $\{e_i\}$ of the T, a data (T, α, ϕ, β) induces the data $(T', \alpha', \phi', \beta')$.

There is the natural ordering of the set

$$\mathcal{S}_{n,m} := \{(T, \alpha, \phi, \beta) | \text{ as above } \#\mathrm{En}_1(T) = m, \#\mathrm{En}_2(T) = n\}.$$

For two oriented 2-tree whose base trees coincide, we define $(T, \alpha, \phi, \beta) \le (T', \alpha', \phi', \beta')$ if $\alpha = \alpha'$, $\phi \le \phi'$ on $\mathrm{En}(T)$, $\phi = \phi'$ on $E(T) - \mathrm{En}(T)$, and the natural maps $\mathrm{St}_2(u, \phi') \to \mathrm{St}_2(u, \phi)$ are order preserving for all vertices u. In general, we define $(T, \alpha, \phi, \beta) \le (T', \alpha', \phi', \beta')$ if T' is obtained by collapsing some edges $\{e_i\}$ of T, and if it holds that $(T', \tilde{\alpha}, \tilde{\phi}, \tilde{\beta}) \le (T', \alpha', \phi', \beta')$, where $(T', \tilde{\alpha}, \tilde{\phi}, \tilde{\beta})$ is the data induced by the (T, α, ϕ, β).

Take an oriented 2-tree (T, α, ϕ, β) of type (n, m). Then we have the subset $C_{(T,\alpha,\phi,\beta)} \subset \bar{C}_{n,m}$. The set $C_{(T,\alpha,\phi,\beta)}$ is isomorphic to the following product:

$$C_{(T,\phi)} := \prod_{\substack{v \in \mathrm{In}(T) \\ \phi(e(v))=2}} C_{\#\mathrm{St}_1(v), \#\mathrm{St}_2(v)} \times \prod_{\substack{v \in \mathrm{In}(T) \\ \phi(e(v))=1}} C_{\#\mathrm{St}_1(u)},$$

where $e(u)$ denotes the edge ending u for any vertex u. If we have that $(T, \alpha, \phi, \beta) \le (T', \alpha', \phi', \beta')$, then the closure of $C_{(T',\alpha',\phi',\beta')}$ contains the set $C_{(T,\alpha,\phi,\beta)}$. We have that $C_{(T,\alpha,\phi,\beta)} \cap C_{(T',\alpha',\phi',\beta')} = \emptyset$, if $(T, \alpha, \phi, \beta) \ne (T', \alpha', \phi', \beta')$.

The collection $\{C_{(T,\alpha,\phi,\beta)}\}$ gives a stratification of the $\bar{C}_{n,m}$.

2.3.2. The stratification of D_l

The stratification of $\bar{C}_{l+1,0}$ is inherited by the D_l, i.e., for any numbered oriented 2-tree T with type $(l + 1, 0)$, we put $D_{lT} := D_l \cap C_T$. As is easily seen, if $D_{lT} \ne \emptyset$, then it holds that $\mathrm{St}(u) = \mathrm{St}_1(u)$ or $\mathrm{St}(u) = \mathrm{St}_2(u)$ for any inner vertex u of T. For such T, we put $\epsilon(u) = i$ if $\mathrm{St}(u) = \mathrm{St}_i(u)$. The D_{lT} is isomorphic to the following product:

$$D_{lT} \simeq \prod_{\substack{u \in \mathrm{In}(T) \\ \epsilon(u) \ge \phi(u)}} \mathring{\Delta}^{\#\mathrm{St}(u)-2} \times \prod_{\substack{u \in \mathrm{In}(T) \\ \epsilon(u) < \phi(u)}} \mathring{\Delta}^{\#\mathrm{St}(u)-1}. \tag{10}$$

We obtain the following equality:

$$\dim D_{lT} = \sum_{\substack{u \in \mathrm{In}(T) \\ \epsilon(u) \le \phi(u)}} (\#\mathrm{St}(u) - 2) + \sum_{\substack{u \in \mathrm{In}(T) \\ \epsilon(u) > \phi(u)}} (\#\mathrm{St}(u) - 1). \tag{11}$$

In particular, the codimension 1 strata correspond to the following type of oriented 2-trees:

- The set of inner vertices of T is $\mathrm{In}(T) = \{\mathrm{root}, u\}$. For all non-minimal edges e, it holds that $\phi(e) = 1$ and that $\phi(e(\mathrm{root})) = 2$. The corresponding stratum is the following:

  $$D_l \cap C_T \simeq \mathring{\Delta}^{l_0-1} \times \mathring{\Delta}^{l_1-2},$$

  where $l_0 = \#\mathrm{St}(\mathrm{root})$, $l_1 = \#\mathrm{St}(u)$ and it holds that $l_0 + l_1 = l + 1$. This type can be seen as the boundary of the $\bar{C}_{0,l+1} \times \mathbb{R}_+$.

- The set of inner vertices of T is $\mathrm{In}(T) = \{\mathrm{root}, u_1, \dots, u_k\}$. There are the edges root $\to u$ connecting root and u. We have that $\mathrm{St}(\mathrm{root}) = \mathrm{St}_2(\mathrm{root})$ and that $\mathrm{St}(u_i) = \mathrm{St}_1(u)$. The corresponding stratum is the following:

$$D_{lT} \simeq C_{0,l_0} \times \prod_{\alpha=1}^{k} \mathring{\Delta}^{l_\alpha - 1},$$

where $l_0 = \#\mathrm{St}(\mathrm{root}),\ l_\alpha = \#\mathrm{St}(u_\alpha)$ and it holds that $\sum l_j = l + k$.

2.3.3. *The stratification of $X^l_{n,m}$*

The stratification of $X^l_{n,m}$ is obtained from that of $\bar{C}_{n,m}$. Hence, we obtain the following proposition.

Proposition 2.1. *The list of the boundary strata with codimension* 1 *of the* $\bar{X}^l_{n,m}$ *is the following type of oriented* 2*-trees* (T, α, ϕ, β):

1. *The set of the inner vertices is* $\{\mathrm{root}, u\}$. *We have that* $\mathrm{St}(u) = \mathrm{St}_1(u)$, *and that* $\phi(u) = 1$. *By the numbering* α, *the set of the end points of the edge in the* $\mathrm{St}(u)$ *contains at most one element of the set* $\{1, \dots, l\}$.
2. *The set of the inner vertices is* $\{\mathrm{root}, u\}$. *We have that* $\phi(u) = 2$. *By the numbering* α, *the set of the end points of the edge in the* $\mathrm{St}_1(u)$ *is contained in the set* $\{l + 1, \dots, n\}$.
3. *The set of inner vertices is* $\{root, u\}$. *We have that* $\mathrm{St}(u) = \mathrm{St}_1(u)$, *and that* $\phi(u) = 1$. *By the numbering* α, *the set of the end points of the edge in the* $\mathrm{St}_1(u)$ *contains at least two elements of the set* $\{1, \dots, l\}$.
4. *The set of inner vertices is* $\{\mathrm{root}, u_1, \dots, u_l\}$. *We have that* $\phi(\mathrm{root} \to u_i) = 2$. *For the edge e ending at an element* $k \in \{1, 2, \dots, l\} \subset \mathrm{En}(T)_2$, *it holds that* $\phi(e) = 1$ *if e starts at the root and that* $\phi(e) = 2$ *if e starts at the* u_i.

2.4. *l-Admissible graphs, $\mathcal{U}_\Gamma$ and the weight*

In the following, we can use the notation in Kontsevich's paper [10].

Definition 2.2. Let Γ be an admissible graph. Note that admissibility implies that there is no edge ending at the start vertex. We denote the set of the first vertices (resp. the second vertices) of Γ by V^1_Γ (resp. V^2_Γ).

We say that Γ is of type (n, m, e) if $\#V^1_\Gamma = n, \#V^2_\Gamma = m, \#E_\Gamma = e$.

For $l = 0, 1, \dots,$ an l-admissible graph Γ is an admissible graph of type (n, m, e) with the decomposition $V^1_\Gamma = V^{1,1}_\Gamma \sqcup V^{1,2}_\Gamma$, such that:

- $V^{1,1}_\Gamma = \{1, \dots, l + 1\}$, $(\#V^{1,1}_\Gamma = l + 1)$;
- for any pair $(i, j) \subset V^{1,1}_\Gamma$, there are no edges connecting i and j.

We denote the set of l-admissible graph of type $(n, m, 2n + m - 2 - l - k)$ by $G^{l,k}_{n,m}$. If $e = 2n + m - 2 - l$, we denote it also by $G^l_{n,m}$. An usual admissible graph is called $\emptyset$-admissible graph.

Let Γ be an l-admissible graph of type $(n, m, 2n+m-2-l)$. We define the linear map $\mathcal{U}_\Gamma$ of the $\otimes^n T_{\mathrm{poly}}[1]$ to the $D_{\mathrm{poly}}[1+l]$. Consider elements

$$\gamma_i = \frac{1}{(k_i+1)!}\sum_{J_i} \mathrm{asgn}(J_i, (v_p))\gamma_{i,J_i} v_{J_i} \in \Lambda^{k_i+1} T_{\mathrm{poly}}\mathfrak{g},$$

where J_i runs through the set of ordered subset of $\{1, \dots, p\}$ with $\# J_i = k_i + 1$, and where we denote $\otimes_{p\in J_i} v_p$ by v_{J_i}. The map $I : E_\Gamma \to \{1, \dots, p\}$ corresponds to the term $\prod \gamma_{i,J_i} \cdot v_{J_i}[-k_i+1]$ in the development of the $\gamma_1[-k_1+1] \otimes \cdots \otimes \gamma_l[-k_l+1]$.

Following the dg-composition rule, we define $\delta(I)$ as the signature given by the dg-symmetric signature along the exchanges of the order

$$\begin{aligned}\prod \gamma_{i,J_i} \cdot v_{J_i}[-k_i+1] \otimes \prod f_p[1] &\mapsto \left(\prod \gamma_{i,J_i} v_{J_i} \otimes f_p\right)\left[\sum(-k_i+1)+m\right]\\ &\mapsto \left(\prod_{i\in V_\Gamma^1}\prod_{e\in \mathrm{St}'(i)} v_{I(e)} \otimes \gamma_{i,J_i}\right) \otimes \left(\prod_{j\in V_\Gamma^2}\prod_{e\in \mathrm{St}'(j)} v_{I(e)} \otimes f_j\right)\left[\sum(-k_i+1)+m\right].\end{aligned} \tag{12}$$

We put for $\gamma_i' = \gamma[-k_i+1] \in T_{\mathrm{poly}}(A^\cdot)[1]$,

$$\begin{aligned}&\mathcal{U}_\Gamma(\gamma_1' \otimes \cdots \otimes \gamma_n')(f_1 \otimes \cdots \otimes f_m)\\ &\quad := \sum_{I\leftrightarrow\{J_i\}} \delta(I)\left(\prod_{i\in V_\Gamma^1}\prod_{e\in \mathrm{St}'(i)} v_{I(e)}\gamma_{i,J_i}\right) \otimes \left(\prod_{j\in V_\Gamma^2}\prod_{e\in \mathrm{St}'(j)} v_{I(e)} f_j\right).\end{aligned} \tag{13}$$

Definition 2.3. Let Γ be an l-admissible graph of type $(n, m, 2n+m-2-l)$. The number W_Γ^l is defined as follows:

$$W_\Gamma^l := \int_{X_{n,m}^l} w_\Gamma,$$

where w_Γ denotes the Kontsevich form (see [10])

$$w_\Gamma := \prod_{k=1}^{n} \frac{1}{(\#\mathrm{Star}(k))!} \frac{1}{(2\pi)^{2n+m-2-l}} \bigwedge_{e\in E_\Gamma} d\phi_e.$$

We define the map $\mathcal{U}_{n,m}^l$ for any natural number l as follows:

$$\mathcal{U}_{n,m}^l := \sum_{\Gamma\in G_{n,m}^l} W_\Gamma^l \cdot \mathcal{U}_\Gamma : \otimes^n(T_{\mathrm{poly}}(\mathbb{R}^d)[1]) \to D_{\mathrm{poly}}(\mathbb{R}^d)[1+l].$$

We put $\mathcal{U}_n^l := \sum_m \mathcal{U}_{n,m}^l$. For $l = \emptyset$, we put $\mathcal{U}^\emptyset := \mathcal{U}^0$.

2.5. The condition of L_∞-ness

We write down the condition for the morphism $\mathcal{U} = (\mathcal{U}_n)$ to be an L_∞ morphism. For a differential graded Lie algebra $\mathfrak{g}^\cdot$, we put $C(\mathfrak{g}^\cdot) := \mathrm{Sym}(\mathfrak{g}^\cdot[1])$. A morphism $\mathcal{U} : C(T_{\mathrm{poly}}) \to C(D_{\mathrm{poly}})$ is said to be an L_∞-morphism if and only if $\mathcal{U} \circ (\frac{1}{2}[,]) = (d + \frac{1}{2}[,]) \circ \mathcal{U}$.

We denote the T_{poly} and the D_{poly} by $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively.

First we consider the following morphisms $[,] : \mathrm{Sym}^{n+1}(\mathfrak{g}_1[1]) \to \mathrm{Sym}^n(\mathfrak{g}_1[1])[1]$. For an element $\gamma \in \mathfrak{g}_1[1]$, the element $\gamma_1 \cdots \gamma_{n+1} \in \mathrm{Sym}^{n+1}(\mathfrak{g}_1[1])$ is mapped as follows:

$$\begin{aligned}\gamma_1 \cdots \gamma_{n+1} &\mapsto \sum_{i<j} \mathrm{sgn}(\kappa_{ij}, (\gamma_p))(-1)^{\sum_{p=1}^{i-1}|\gamma_p|} \\ &\quad (\gamma_1 \cdots \gamma_{i-1} \cdot [,][-1](\gamma_i \cdot \gamma_j) \cdots \gamma_{n+1})[1] \\ &\quad \mapsto \sum_{i<j} \mathrm{sgn}(\kappa_{ij}, \gamma_p)(-1)^{|\gamma_i[-1]| + \sum_{k \le i-1}|\gamma_k|} \\ &\quad (\gamma_1 \cdots \gamma_{i-1}[\gamma_i[-1], \gamma_j[-1]][1] \cdots \gamma_{n+1})[1]. \end{aligned} \tag{14}$$

Hence, we have the following equalities:

$$\begin{aligned}\mathcal{U}_n\left(\frac{1}{2}[,](\gamma_1 \cdots \gamma_{n+1})\right) &= \sum_{i<j} \mathrm{sgn}(\kappa_{ij}, \gamma_p)(-1)^{|\gamma_i[-1]| + \sum_{k \le i-1}|\gamma_k|} \mathcal{U}_n(\gamma_1 \cdots \gamma_{i-1} \\ &\quad \cdot [\gamma_i[-1], \gamma_j[-1]][1] \cdots \gamma_{n+1})[1]. \end{aligned} \tag{15}$$

By the definition of the bracket, the formula can be rewritten as follows:

$$\begin{aligned}&\sum_{i \ne j} \mathrm{sgn}(\kappa_{ij}, \gamma_p)(-1)^{|\gamma_i[-1]| + \sum_{k \le i-1}|\gamma_k|} \mathcal{U}_n(\gamma_1 \cdots \gamma_{i-1} \cdot (\gamma_i[-1] \\ &\quad \bullet \gamma_j[-1])[1] \cdots \gamma_{n+1})[1]. \end{aligned} \tag{16}$$

Next we consider the following morphisms:

$$\begin{aligned}\frac{1}{2}[,] \circ \mathcal{U} : \mathrm{Sym}^{n+1}(\mathfrak{g}_1[1]) &\to \bigoplus_{k+l=n+1} \mathrm{Sym}^k(\mathfrak{g}_1[1]) \otimes \mathrm{Sym}^l(\mathfrak{g}_1[1]) \\ &\to \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1] \to \mathfrak{g}_2[2]. \end{aligned} \tag{17}$$

Then an element $\gamma_1 \cdots \gamma_{n+1}$ is mapped as follows:

$$\begin{aligned}\gamma_1 \cdots \gamma_{n+1} &\mapsto \sum_{k+l=n+1} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma, (\gamma_p))\gamma_{\sigma_1} \cdots \gamma_{\sigma_k} \otimes \gamma_{\sigma_{k+1}} \cdots \gamma_{\sigma_{n+1}} \\ &\mapsto \sum_{\substack{k+l=n+1 \\ k,l \ge 1}} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma, (\gamma_p))\mathcal{U}_k(\gamma_{\sigma_1} \cdots \gamma_{\sigma_k}) \otimes \mathcal{U}_l(\gamma_{\sigma_{k+1}} \cdots \gamma_{\sigma_{n+1}}) \\ &\mapsto \frac{1}{2} \sum_{\substack{k+l=n+1 \\ k,l \ge 1}} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma, (\gamma_p))(-1)^{|\mathcal{U}_k(\gamma_{\sigma_1} \cdots \gamma_{\sigma_k})[-1]|} \\ &\quad \times [\mathcal{U}_k(\gamma_{\sigma_1} \cdots \gamma_{\sigma_k})[-1], \mathcal{U}_l(\gamma_{\sigma_{k+1}} \cdots \gamma_{\sigma_{n+1}})[-1]][2]. \end{aligned} \tag{18}$$

We put the differential of D_{poly} as follows: $dx = [m, x] = -(-1)^{|x|}[x, m]$. Then we have $d(x[1]) = -[m, x][1] = (-1)^{|x|}[x, m][1]$. Hence $(d + \frac{1}{2}[\,,\,]) \circ \mathcal{U}(\gamma_1 \cdots \gamma_{n+1})$ is described as follows:

$$\frac{1}{2}\sum_{k+l=n+1}\frac{1}{k!l!}\sum_{\sigma\in\Sigma_{n+1}}\mathrm{sgn}(\sigma, (\gamma_p))(-1)^{|\mathcal{U}_k(\gamma_{\sigma_1}\cdots\gamma_{\sigma_k})[-1]|} \times[\mathcal{U}_k(\gamma_{\sigma_1}\cdots\gamma_{\sigma_k})[-1], \mathcal{U}_l(\gamma_{\sigma_{k+1}}\cdots\gamma_{\sigma_{n+1}})[-1]][2]. \tag{19}$$

By the definition of the brackets (6) and (8), we can rewrite it as follows:

$$\sum_{\substack{k+l=n+1\\ k,l\geq 0}}\frac{1}{k!l!}\sum_{\sigma\in\Sigma_{n+1}}\mathrm{sgn}(\sigma, (\gamma_p[1]))(-1)^{rs+|\mathcal{U}_k(\gamma_{\sigma_1}\cdots\gamma_{\sigma_k})[-1]|} \times(\mathcal{U}_k(\gamma_{\sigma_1}\cdots\gamma_{\sigma_k})[-1]\circ\mathcal{U}_l(\gamma_{\sigma_{k+1}}\cdots\gamma_{\sigma_{n+1}})[-1])[2], \tag{20}$$

where we put $r = |\mathcal{U}_k(\gamma_{\sigma_1}\cdots\gamma_{\sigma_k})[-1]|_2$, $s = |\mathcal{U}_l(\gamma_{\sigma_{k+1}}\cdots\gamma_{\sigma_{n+1}})[-1]|_2$.

Therefore the condition for $\mathcal{U}$ to be L_∞ is the following:

$$\text{Formula (16)} = \text{Formula (20)}.$$

3. The compatibility of the cup products

Let $\bar{\Gamma}$ be an l-admissible graph of type $(n+1, m, 2n+m-1-l)$. We rewrite the formula obtained from the Stokes theorem for the integration of the $dw_{\bar{\Gamma}}$ on the $X^l_{n+1,m}$,

$$0 = \int_{X^l_{n+1,m}} dw_{\bar{\Gamma}} = \sum_T \int_{\partial_T X^l_{n+1,m}} w_{\bar{\Gamma}},$$

where T runs through the list in the previous section.

3.1. The types which contribute non-trivially

By the same discussion as that in [10], using the lemma of Kontsevich, we obtain the following list of the oriented 2-trees $T = (T, \alpha, \phi, \beta)$ of type (n, m) such that $\int_{\partial_T X^l_{n+1,m}} w_{\bar{\Gamma}}$ does not vanish:

Type I(i, j) T is of type I such that we have $\#\mathrm{St}(u) = 1$ and that $\alpha(u)(\mathrm{St}(u)) = \{i, j\}$.
Type II(i) T is of type II such that $\beta(\mathrm{root})(\mathrm{root} \to u) = i$.
Type III(i) T is of type III such that we have $\#\mathrm{St}(u)=2$ and that $\alpha(u)(\mathrm{St}(u))=\{i, i+1\}$.
Type IV T is of type IV.

We describe the list of the graphs obtained in the cases above.

Proposition 3.1. *The list of the graphs which must be considered is the following:*

I(i, j) Γ: *an l-admissible graph of type* $(n, m, 2n+m-2-l)$ *obtained by collapsing the edge* $i \to j$ *to a point ij.*

II(i) Γ_0 *and* Γ_1: *they satisfy the following*:

- Γ_0 *is an l-admissible graph of type* $(n_0, m_0, 2n_0 + m_0 - 2 - l)$.
- Γ_1 *is a* 0*-admissible graph of type* $(n_1, m_1, 2n_1 + m_1 - 2)$.
- Γ_1 *is a subgraph of* $\bar{\Gamma}$ *whose second vertices are* $\{i, \dots, i + m_1 - 1\}$. Γ_0 *is obtained by collapsing* Γ_1 *to a point.*

We describe the situation as $\Gamma_0 \Rightarrow_i \Gamma_1$.

III (i) Γ: *An* $(l-1)$*-admissible graph of type* $(n, m, 2n+m-1-l)$ *obtained by collapsing two points to a point from two vertices* $i, i+1$ *of the type* $(1, 1)$.

IV Γ_0 *and* $\Gamma_\alpha (i_\alpha = 1, \dots, k)$, *where* Γ_0 *is a* 0*-admissible graph of type* $(n_0, m_0, 2n_0 + m_0 - 2)$. Γ_α *is* l_α*-graph of type* $(n_\alpha, m_\alpha, 2n_\alpha + m_\alpha - 2 - l_\alpha)$. *Moreover, it holds that* $\sum n_\alpha = n + 1, \sum m_\alpha = m + k \sum (l_\alpha + 1) = l + 1$.

We describe the situation as $\Gamma_0 \Rightarrow_{j_\alpha} \Gamma_\alpha$.

3.2. The contributions

Let M be a manifold with the boundary ∂M. We determine the orientation of the boundary as follows:

$$(\text{The orientation of } M) = (\text{The inner normal vector}) \times (\text{The orientation of } \partial M).$$

Let γ_i be homogeneous elements of $T_{\text{poly}}[1]$ and f_i homogeneous elements of $A^{\cdot}[1]$ in the following. If homogeneous $\gamma_i = (\sum \gamma_{iI} v_I)[-k_i + 1] \in \Lambda^{k_i+1}\mathfrak{g}^{\cdot}$, we put $|\gamma_i|_1 = {}^{\circ}(\gamma_{iI} v_I)$, $|\gamma_i|_2 = -k_i + 1$. It holds that $|\gamma_i| = |\gamma_i|_1 + |\gamma_i|_2$ by definition.

In the following, we will rewrite the following by using Stokes formula:

$$\sum_{\bar{\Gamma} \in G^{l,1}_{n+1,m}} \sum_T \left(\int_{\partial_T X^l_{n+1,m}} w_{\bar{\Gamma}} \right) \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1}).$$

Note that the $X^l_{n,m}$ can be regarded as polyhedron locally, and hence we can use the Stokes formula. In the following, we also denote the components of the boundary of the $X^l_{n+1,m}$ by the $\partial_\Gamma X^l_{n+1,m}$, if the T and the $\bar{\Gamma}$ give the graph Γ.

Definition 3.1. Let V be a graded vector space. Consider homogeneous elements $a_i \in V$. In the following, the transition from $a_1[l_1] \otimes \cdots \otimes a_m[l_m]$ to $a_{\sigma(1)}[l_{\sigma(1)}] \otimes \cdots \otimes a_{\sigma(n)}[l_{\sigma(n)}]$ means that the composition of the following exchange of the order:

$$a_1[l_1] \otimes \cdots \otimes a_m[l_m] \mapsto a_1 \otimes \cdots \otimes a_m \left[\sum l_p\right] \mapsto a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \left[\sum l_p\right]$$
$$\mapsto a_{\sigma(1)}[l_{\sigma(1)}] \otimes \cdots \otimes a_{\sigma(n)}[l_{\sigma(n)}]. \quad (21)$$

Notice that we do not replace the order of the shift.

We use the following lemma to see the signatures, which can be shown by a direct calculation.

Lemma 3.1. *The dg-symmetric signature $t(\sigma, (a_i[l_i]))$ of the transition above is the following in the $\mathbb{Z}/2\mathbb{Z}$:*

$$t(\sigma, (a_i[l_i]))' := \mathrm{sgn}(\sigma, (a_i[l_i]))' - \mathrm{sgn}(\sigma, (|a_i[l_i]|_2))'$$
$$= \mathrm{asgn}(\sigma, (a_i[l_i]))' - \mathrm{asgn}(\sigma, (|a_i[l_i]|_2))'.$$

3.2.1. Case $I(i, j)$

Let Γ be an l-admissible graph of type $(n, m, 2n + m - 2 - l)$. We denote by $S(\Gamma, i, j)$ the set of l-admissible graphs $\bar{\Gamma}$ of type $(n + 1, m, 2n + m - 1 - l)$ with the following conditions:

- The edge $i \to j$ exists for $i < j$.
- The graph Γ is obtained from the $\bar{\Gamma}$ by collapsing the edge $i \to j$ to the vertex *ij*, whose number is i.
- We denote the numbering of the first vertices of the Γ by n_Γ, i.e., $n_\Gamma : V^1_\Gamma \xrightarrow{\sim} \{1, \dots, n\}$. Then we have the following:

  $$n_{\bar{\Gamma}}(p) = n_\Gamma(p) \quad \text{if } \ n_\Gamma(p) < j, \qquad n_{\bar{\Gamma}}(p) = n_\Gamma(p) + 1 \quad \text{if } \ n_\Gamma(p) \geq j,$$

  under the isomorphism $V^1_{\bar{\Gamma}} \simeq V^1_\Gamma$.
- The ordering of the $\mathrm{St}(p, \bar{\Gamma})$ is same as that of the $\mathrm{St}(p, \Gamma)$ if $n_\Gamma(p) \neq i, j$. The ordering of the $\mathrm{St}(i, \bar{\Gamma}) - \{i \to j\}$ and the $\mathrm{St}(j, \bar{\Gamma})$ is the restriction of that of the $\mathrm{St}(ij)$.

Then it holds that

$$(-1)^{|\gamma_i|-1+\sum_{p<i}|\gamma_p|_1} t(\kappa_{i,j}, (\gamma_p)) \mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes (\gamma_i[-1] \bullet \gamma_j[-1])[1] \otimes \cdots \otimes \gamma_{n+1})$$
$$= \frac{1}{\#\mathrm{St}(i)} \sum_{\bar{\Gamma} \in S(\Gamma, i, j)} \mathrm{asgn}(\tau) \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1}), \tag{22}$$

where τ denotes the permutation of the order

$$\{i \to j\} \sqcup \mathrm{St}(ij; \bar{\Gamma}) \to \mathrm{St}(i) \cup \mathrm{St}(j).$$

The term $\#\mathrm{St}(i)^{-1}$ appears because of the ambiguity of the order of the edge $i \to j$ in $\mathrm{St}(i)$. On the other hand, we have the equality

$$\int_{\partial_\Gamma X^l_{n+1,m}} \bigwedge_{e \in E_{\bar{\Gamma}}} d\phi_e$$
$$= \int_{\partial_\Gamma X^l_{n+1,m}} \mathrm{sgn}(\kappa_{ij}, (|\gamma_p|_2)) \cdot \mathrm{asgn}(\tau) \cdot (-1)^{\sum_{p \leq i-1}(|\gamma_p|_2)} d\phi_{e_{ij}} \wedge \bigwedge_{e \in E_\Gamma} d\phi_e$$
$$= (-1)^{\sum_{p \leq i-1}(|\gamma_p|_2)} 2\pi \int_{X^l_{n,m}} \mathrm{sgn}(\kappa_{ij}, (|\gamma_p|_2)) \cdot \mathrm{asgn}(\tau) \bigwedge_{e \in E_\Gamma} d\phi_e. \tag{23}$$

Hence, we obtain the following equality:

$$W_\Gamma = (-1)^{\sum_{p \leq i-1}|\gamma_p|_2} \mathrm{asgn}(\tau) \mathrm{sgn}(\kappa_{ij}, (|\gamma_p|_2)) \frac{\#\mathrm{St}(i)!\#\mathrm{St}(j)!}{(\#\mathrm{St}(i) + \#\mathrm{St}(j) - 1)!} \int_{\partial_\Gamma X^l_{n,m}} w_{\bar{\Gamma}}.$$

Therefore, we obtain the following equality:

$$
\begin{aligned}
&(-1)^{\sum_{p\le i-1}|\gamma_p|+|\gamma_i|-1}\mathrm{sgn}(\kappa_{ij},(\gamma_p))W_\Gamma\\
&\quad\cdot\mathcal{U}_\Gamma(\gamma_1\otimes\cdots\otimes(\gamma_i[-1]\bullet\gamma_j[-1])[1]\otimes\cdots\otimes\gamma_{n+1})\\
&\quad=\sum_{\bar\Gamma\in S(\Gamma,i,j)}\frac{(\#\mathrm{St}(i)-1)!\#\mathrm{St}(j)!}{(\#\mathrm{St}(i)+\#\mathrm{St}(j)-1)!}\left(\int_{\partial_\Gamma X^l_{n,m}}w_{\bar\Gamma}\right)\cdot\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{n+1}).
\end{aligned}\tag{24}
$$

As is easily seen, the number of graphs Γ such that $\bar\Gamma\in S(\Gamma,i,j)$ is $(\#\mathrm{St}(i)+\#\mathrm{St}(j)-1)!/(\#\mathrm{St}(i)-1)!\#\mathrm{St}(j)!$. Hence, we have the following formula:

$$
\begin{aligned}
&\sum_{\Gamma\in G^l_{n,m}}(-1)^{\sum_{k\le i-1}|\gamma_k|+|\gamma_i[-1]|}\mathrm{sgn}(\kappa_{ij},(\gamma_p))\cdot W_\Gamma\\
&\quad\cdot\mathcal{U}_\Gamma(\gamma_1\otimes\cdots\otimes(\gamma_i[-1]\bullet\gamma_j[-1])[1]\otimes\cdots\otimes\gamma_{n+1})\\
&=\sum_{T,\text{type I}(i,j)}\sum_{\bar\Gamma\in G^{l,1}_{n+1,m}(i\to j)}\left(\int_{\partial_T X^l_{n+1,m}}w_{\bar\Gamma}\right)\cdot\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{n+1}),
\end{aligned}\tag{25}
$$

where $G^{l,1}_{n+1,m}(i\to j)$ is the set of graphs in $G^{l,1}_{n+1,m}$ such that it contains the edge $i\to j$.

The case $i>j$ is similar.

3.2.2. *Case II*(i)

Let Γ_0 be an l-admissible graph of type $(n_0,m_0,2n_0+m_0-2-l)$ and Γ_1 be an $\emptyset$-admissible graph of type $(n_1,m_1,2n_1+m_1-2)$. We have the equalities $n_0+n_1=n+1, m_0+m_1=m+1$. We denote by $S(\Gamma_0,\Gamma_1,i)$ the set of unnumbered l-admissible graph such that it is obtained from the graphs Γ_0 and Γ_1 by removing the vertex i of the second type and by connecting the edges in the set $\mathrm{St}'(i)$ to the vertices of the graph Γ_1. The terminology "unnumbered" means that $\bar\Gamma$ is not given the numbering of the set of the vertices of $(1,2)$-type. Notice that $V_\Gamma=V_{\Gamma_0}\cup V_{\Gamma_1}$. For any vertex p of the graph $\bar\Gamma$, the ordering of the set $\mathrm{St}(p;\bar\Gamma)$ are determined by that of the $\mathrm{St}(p';\Gamma)$, where p' denotes the corresponding vertex in the Γ.

For any vertex $v\in V^1_{\Gamma_0}\sqcup V^1_{\Gamma_1}$, we denote the γ_i on the vertex v by η_v.

Lemma 3.2. *We have the following equality*:

$$
\begin{aligned}
&\delta_1\cdot t(\sigma,(\gamma_p))\mathcal{U}_{\Gamma_0}\left(\bigotimes_{v\in V^1_{\Gamma_0}}\eta_v\right)\left(f_1\otimes\cdots\right.\\
&\quad\left.\otimes\,\mathcal{U}_{\Gamma_1}\left(\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\right)[-1](f_i\otimes\cdots\otimes f_{i+m_1-1})\otimes\cdots\otimes f_m\right)\\
&\quad=\sum_{\bar\Gamma\in S(\Gamma_0,\Gamma_1,i)}\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}\otimes\gamma_{l+2}\otimes\cdots\otimes\gamma_{n+1}),
\end{aligned}\tag{26}
$$

where we put as follows:

$$\delta_1' \equiv \left|\mathcal{U}_{\Gamma_1}\left(\bigotimes_{v\in V_{\Gamma_1}}\eta_v\right)[-1]\right|\cdot\sum^{i-1}|f_q|+\sum_{v\in V^1_{\Gamma_0}}|\eta_v|_1+(i-1)(m_1+1)\in\mathbb{Z}/2\mathbb{Z},$$

and where we give the numbering of the set $V^{1,2}_{\bar{\Gamma}}$ *by the* $\{\gamma_p\}$, *i.e., the number of the v equals p if the* γ_p *is on the v, and where* σ *is the permutation from the* $\{\gamma_p\}$ *to the* $\{\eta_v|v\in V^1_{\Gamma_0}\}\sqcup\{\eta_v|v\in V^1_{\Gamma_1}\}$.

Proof. We have to see the dg-symmetric signature of the following transition:

$$\begin{aligned}&\bigotimes_{v\in V^1_{\Gamma_0}}\eta_v\otimes\left(\bigotimes_{q=1}^{i-1}f_q\right)\otimes\left\{\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\otimes\left(\bigotimes_{q=i}^{i+m_1-1}f_q\right)\right\}\left[\sum_{v\in V^1_{\Gamma_1}}|\eta_v|_2-m_1+1\right]\otimes\cdots\\ &\quad\mapsto\left(\otimes\gamma_j\otimes\left(\bigotimes_{q=1}^{m}f_q\right)\right)\left[\sum_{v\in V^1_{\Gamma_1}}|\eta_v|_2-m_1+1\right].\end{aligned}\tag{27}$$

We have the equality $\sum_{v\in V^1_{\Gamma_1}}|\eta_v|_2-m_1+1=2n_1+m_1-2-2n_1-m_1+1=-1$. Hence the dg-symmetric signature is as follows:

$$\sum_{v\in V^1_{\Gamma_1}}|\eta_v|\cdot\sum^{i-1}|f_q|-(i-1)\cdot\sum_{v\in V^1_{\Gamma_1}}|\eta_v|_2+\sum_{v\in V^1_{\Gamma_0}}|\eta_v|_1+\sum^{i-1}|f_q|_1\equiv\delta'.\tag{28}$$

Hence we are done. □

On the other hand, we must determine the signature of the integral, i.e., the orientation of the boundary. We see the coordinate transformation around $\partial_{\Gamma_0\Rightarrow\Gamma_1}\bar{X}^l_{n+1,m}$ as follows:

$$\begin{aligned}&(z_1,\dots,z_{n+1};p_1,\dots,p_m)\sim(z_1,\dots,z_{l+1},z_{\sigma_{l+2}},\dots,z_{\sigma_{n_0}};p_1,\dots,p_{i-1},\\ &\quad p_{i+m_1},\dots,p_m;z_{\sigma_{n_0+1}},\dots,z_{\sigma_{n+1}};p_i,p_{i+1},\dots,p_{i+m_1-1})\times(-1)^{(m-(i+m_1-1))m_1}\\ &\quad\sim(\tau,z_1,\dots,z_{l+1};z_{\sigma_{l+2}},\dots,z_{\sigma_{n_0}};p_1,\dots,p_{i-1},p_{i+m_1},\dots,p_m)\\ &\quad\times(z_{\sigma_{n_0+1}},\dots,z_{\sigma_{n+1}};p_i,p_{i+1},\dots,p_{i+m_2-1})\times(-1)^{(m-(i+m_1-1))m_1},\end{aligned}$$

where τ denotes the coordinate of the 'center point' of $(z_{\sigma_{n_0+1}},\dots,z_{\sigma_{n+1}};p_i,p_{i+1},\dots,p_{i+m_2-1})$. Furthermore, it is equivalent to the following orientation:

$$\begin{aligned}&\sim(\text{inner normal},z_1,\dots,z_{l+1},z_{\sigma_{l+2}},\dots,z_{\sigma_{n_0}};p_1,\dots,p_{i-1},\bar{p}_i,p_{i+m_1},\dots,p_m)\\ &\quad\times(z_{\sigma_{n_0+1}},\dots,z_{\sigma_{n+1}};p_i,\dots,p_{i+m_2-1})\times\delta_2,\end{aligned}$$

where we put $\delta_2'=(m_1+1)(m_0+1)+(i-1)(m_1+1)+m_0+l$.

Hence, we obtain the following equality:

$$\delta\cdot\mathrm{sgn}(\sigma,(\gamma_p))W_{\Gamma_0}W_{\Gamma_1}\cdot\mathcal{U}^l_{\Gamma_0}\left(\bigotimes_{v\in V^1_{\Gamma_0}}\eta_v\right)[-l-1]\circ_i\mathcal{U}^0_{\Gamma_1}\left(\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\right)[-1]$$
$$=\sum\frac{1}{(n-l)!}\int_{\partial_{\Gamma_0\Rightarrow\Gamma_1}X^l_{n+1,m}}w_\Gamma\cdot\mathcal{U}_{\bar\Gamma}(\gamma'_1\otimes\cdots\otimes\gamma'_{n+1}), \tag{29}$$

where we put as follows:

$$\delta'\equiv-\sum_{v\in V^1_{\Gamma_0}}|\eta_v|_1+m_0+l+(m_1+1)(m_0+1)$$
$$\equiv\left|\mathcal{U}_{\Gamma_0}\left(\bigotimes_{v\in V^1_{\Gamma_0}}\eta_v\right)[-l-1]\right|+l+1+(m_0+1)(m_1+1).$$

The numbering of $V^{1,2}_{\bar\Gamma}$ is given as above. Notice that the permutation $\sigma\in\Sigma_{n-l}$ corresponds to the numbering of the set $V^{1,2}_{\bar\Gamma}$. Notice also that the numberings of the graphs Γ_0 and Γ_1 do not effect the right-hand side. Therefore, we arrived at the following equality:

$$\frac{1}{(n_0-l-1)!n_1!}$$
$$\times\sum_{\sigma\in\Sigma_{n-l}}\mathrm{sgn}(\sigma,(\gamma_p))\delta\mathcal{U}^l_{n_0,m_0}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}\otimes\gamma_{\sigma_{l+2}}\otimes\cdots\otimes\gamma_{\sigma_{n_0}})[-l-1]$$
$$\circ_i\mathcal{U}_{n_1,m_1}(\gamma_{\sigma_{n_0+1}}\otimes\cdots\otimes\gamma_{\sigma_{n+1}})[-1]$$
$$=\sum_{T,\text{type II}(i)}\ \sum_{\bar\Gamma\in G^{l,1}_{n+1,m}}\int_{\partial_TX^l_{n+1,m}}w_{\bar\Gamma}\cdot\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{n+1}). \tag{30}$$

3.2.3. *Case III(i)*

Let Γ be an $(l-1)$-admissible graph of type $(n,m,2n+m-1-l)$. Assume that $i<l$. We denote by $S'(\Gamma,i)$ the set of an l-admissible graphs $\bar\Gamma$ of type $(n+1,m,2n-m-1-l)$ with the following conditions:

- The graph Γ is obtained from the $\bar\Gamma$ by collapsing two vertices i and $i+1$ to one edge $\bar i$, whose number is i.
- The numbering of the $V_{\bar\Gamma}$ is $n_{\bar\Gamma}(p)=n_\Gamma(p)$ if $n_\Gamma(p)<i$, $n_{\bar\Gamma}(p)=n_\Gamma(p)+1$ if $n_\Gamma(p)>i$ under the isomorphism $V^1_\Gamma-\{\bar i\}\simeq V^1_{\bar\Gamma}-\{i,i+1\}$.
- The ordering of the set $\mathrm{St}(i)$ and $\mathrm{St}(i+1)$ are the restrictions of that of the $\mathrm{St}(\bar i)$. The orderings of the $\mathrm{St}(p)$ for any vertices $p\neq i,\ i+1$ are inherited from the corresponding vertices of the Γ.

We have the following equality:

$$\mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes (\gamma_i[-2] \wedge \gamma_{i+1}[-2])[2] \otimes \cdots \otimes \gamma_{n+1}) = \sum_{\bar{\Gamma} \in G^{l,1}_{n+1}} \mathrm{sgn}(\tau) \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1}), \tag{31}$$

where τ is the permutation of the order $\mathrm{St}(\bar{i}) \to \mathrm{St}(i) \sqcup \mathrm{St}(i+1)$. Moreover, we obtain the following equality:

$$\frac{\#\mathrm{St}(i)!\#\mathrm{St}(i+1)!}{\#\mathrm{St}(\bar{i})!}(-1)^i W_\Gamma \cdot \mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes (\gamma_i[-2] \wedge \gamma_{i+1}[-2])[2] \otimes \cdots \otimes \gamma_{n+1}) = \sum \int_{\partial_\Gamma X^l_{n+1,m}} w_{\bar{\Gamma}} \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1}). \tag{32}$$

As is easily seen, the number of graphs Γ such that the $\bar{\Gamma}$ is contained in the $S'(\Gamma, i)$ is $\#\mathrm{St}(\bar{i})!/\#\mathrm{St}(i)!\#\mathrm{St}(i+1)!$. Hence, we arrive at the following equality:

$$-\sum_{\Gamma \in G^l_{n,m}} (-1)^{i-1} W_\Gamma \mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes (\gamma_i[-2] \wedge \gamma_{i+1}[-2])[2] \otimes \cdots \otimes \gamma_{n+1}) = \sum_{T,\text{type III}(i)} \int_{\partial_T X^{l-1}_{n+1,m}} w_{\bar{\Gamma}} \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1}). \tag{33}$$

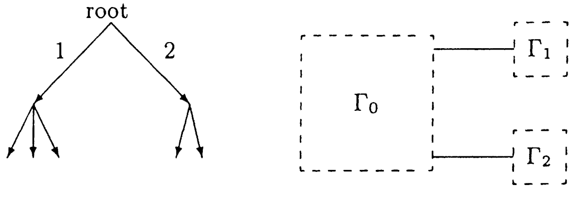
3.2.4. Case IV

In this case, the signature is slightly complicated. We give the proof of the case needed when $A = C^\infty(\mathbb{R}^d)$. Then we state the result in the general case:

The case $l = 0$. Similar to case II.

The case $l \geq 1$. We only write the results in the case $m_0 = 2$. It is divided into three cases:

1. We consider the situation corresponding to the following tree:



where the number is the value of $\beta(\text{root})$. Let Γ_0 be a 0-admissible graph of type $(n_0, 2, 2n_0)$, and $\Gamma_\alpha(\alpha = 1, 2)$ be l_α-admissible graphs of type $(n_\alpha, m_\alpha, 2n_\alpha + m_\alpha - 2 - l_\alpha)$. We have $\sum n_\alpha = n + 1, m_1 + m_2 = m$ and $l_\alpha \neq \emptyset$ for $\alpha = 1, 2$.

We denote by $S(\Gamma_0, \Gamma_1, \Gamma_2)$ the set of unnumbered l-admissible graphs obtained from Γ_α by removing the vertices $1, 2 \in V^2_{\Gamma_0}$ and connecting edges in St(1) and St(2) to the vertices of Γ_1, Γ_2. The orderings of stars are inherited from those of Γ_α.

We have the following equality:

$$\delta_1 \cdot t(\sigma, (\gamma_p)) \times \mathcal{U}_{\Gamma_0}[-1]\left(\bigotimes_{v\in\Gamma_0}\eta_v\right)\left(\mathcal{U}_{\Gamma_1}\left(\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\right)[-l_1-1](f_1\otimes\cdots\otimes f_{m_1})\right.$$
$$\left.\otimes\,\mathcal{U}_{\Gamma_2}\left(\bigotimes_{v\in V^1_{\Gamma_2}}\eta_v\right)[-l_2-1](f_{m_1+1}\otimes\cdots\otimes f_m)\right)$$
$$= \sum_{\bar{\Gamma}\in S(\Gamma_k)}\mathcal{U}_{\bar{\Gamma}}(\gamma_1\otimes\cdots\otimes\gamma_{n+1})(f_1\otimes\cdots\otimes f_m). \tag{34}$$

where we put as follows:

$$\delta_1' \equiv \sum_{v\in V^1_{\Gamma_2}}|\eta_v|_1\sum_{i=1}^{m}|f_i| + \sum_{q=1}^{2}(l_q+1)\sum_{t=1}^{q-1}\sum_{v\in V^1_{\Gamma_t}}|\eta_v|_1 + (m_2+1)\sum_{i=1}^{m_1}|f_i|_1 \pmod 2.$$

The numbering of $\bar{\Gamma}$ is thought by the same way as that in case II.

We must determine the signature of the integral. We put $z_i = r_i + \sqrt{-1}t\,(i = 1,\dots,l+1,\ r_1 = 0, r_{l+1} = 1)$. We consider the following coordinate transformations:

$$\begin{aligned}
&(t, r_1 = 0, r_2,\dots,r_l, r_{l+1} = 1, z_{l+2},\dots,z_n; p_1,\dots,p_m)\\
&\quad\simeq (t, z_{\sigma_{l+2}},\dots,z_{\sigma_{n_0+l+1}}; 0, 1, r_1 = 0, r_2,\dots,r_{l_1+1}, z_{\sigma_{n_0+l+2}},\dots,z_{\sigma_{n_0+n_1+l_2+1}};\\
&\qquad p_1,\dots,p_{m_1}; r_{l_1+2},\dots,r_l, r_{l+1} = 1, z_{\sigma_{n_0+n_1+l_2+2}},\dots,z_{\sigma_{n+1}};\\
&\qquad p_{m_1+1},\dots,p_m) \times (-1)^{l_2m_1}.
\end{aligned}$$

We put $s_1 = r_{l_1+1} - r_1 = r_{l_1+1}$ and $s_2 = r_{l+1} - r_{l_1+2} = 1 - r_{l_1+2}$. Then it is transformed to the following:

$$\begin{aligned}
&\simeq \left(t, z_{\sigma_{l+2}},\dots,z_{\sigma_{n_0+l+1}}; 0, 1; s_1, r_1 = 0, \frac{r_2}{s_1},\dots,\frac{r_{l_1}}{s_1}, \frac{r_{l_1+1}}{s_1} = 1,\right.\\
&\quad \frac{z_{\sigma_{n_0+l+2}}}{s_1},\dots,\frac{z_{\sigma_{n_0+n_1+l_2+1}}}{s_1}; \frac{p_1}{s_1},\dots,\frac{p_{m_1}}{s_1}; s_2, 0, r_{l_1+2},\dots,r_l, r_{l+1} = 1,\\
&\quad \left. z_{\sigma_{n_0+n_1+l_2+2}},\dots,z_{\sigma_{n+1}}; p_{m_1+1},\dots,p_m\right) \times (-1)^{l_1+l_2m_1},
\end{aligned}$$

where we put $z = (1/s_2)(z-1)+1$. Notice the following equality:

$$l_1 + l_2m_1 \equiv l_1 + 1 + (l_2+1)m_1 + (m_2-1)m_1 + 2 + (m_1-1) + (m_2-1)m_1.$$

Hence, we obtain the following:

$$\begin{aligned}
&\delta \cdot \mathrm{sgn}(\sigma, (\gamma_p)) W_{\Gamma_0} W_{\Gamma_1} W_{\Gamma_2} \\
&\quad \times \mathcal{U}_{\Gamma_0}\left(\bigotimes_{v \in V^1_{\Gamma_0}} \eta_v\right)\left(\mathcal{U}_{\Gamma_1}\left(\bigotimes_{v \in V^1_{\Gamma_1}} \eta_v\right)[-l_1 - 1](f_1 \otimes \cdots \otimes f_{m_1})\right. \\
&\quad \left. \otimes \mathcal{U}_{\Gamma_2}\left(\bigotimes_{v \in V^1_{\Gamma_2}} \eta_v\right)[-l_2 - 1](f_{m_1+1} \otimes \cdots \otimes f_m)\right) \\
&= \sum_{\bar{\Gamma} \in S(\Gamma_i)} \int_{\partial_{\Gamma_0 \Rightarrow \Gamma_i} X^l_{n+1,m}} w_{\bar{\Gamma}} \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1})(f_1 \otimes \cdots \otimes f_m),
\end{aligned} \tag{35}$$

where we put as follows:

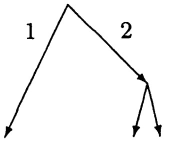
$$\begin{aligned}
\delta' &\equiv \sum_{q=1}^{2}\left(\sum_{p=1}^{j_q - 1} |f_p|\right)\left|\mathcal{U}_{\Gamma_q}\left(\bigotimes_{v \in V^1_{\Gamma_q}} \eta_v\right)[l_q - 1]\right| \\
&\quad + \sum_{q=1}^{2}(l_q + 1)\sum_{t=0}^{q-1}\left|\mathcal{U}_{\Gamma_t}\left(\bigotimes_{v \in V^1_{\Gamma_t}} \eta_v\right)[l_t - 1]\right| \\
&\quad + (m_1 - 1) + (m_2 - 1)(m_1) \pmod 2.
\end{aligned} \tag{36}$$

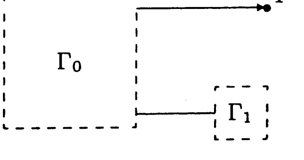
As in case II, σ corresponds to the numberings of $\bar{\Gamma}$, and the numberings of Γ_1 and the Γ_2 do not effect the right-hand side.

Hence, we obtain the following equality:

$$\begin{aligned}
&\frac{1}{(n_0 - l - 1)! n_1! n_2!} \sum_{\sigma \in \Sigma_{n-l}} \delta_2 \, \mathrm{asgn}(\sigma, (\gamma_p)) \mathcal{U}_{n_0,2}(\gamma_{\sigma_{l+2}} \otimes \cdots \otimes \gamma_{\sigma_{l+n_0+1}}) \\
&\quad \times (\mathcal{U}^{l_1}(\gamma_1 \otimes \cdots \otimes \gamma_{l_1+1} \otimes \gamma_{\sigma_{l+n_0+2}} \otimes \cdots \otimes \gamma_{\sigma_{n_0+n_1+l_2+1}}) \\
&\quad \times [-l_1 - 1](f_1 \otimes \cdots \otimes f_{m_1}) \otimes \mathcal{U}^{l_1}_{n_1,m_1} \\
&\quad \times (\gamma_{l_1+2} \otimes \cdots \otimes \gamma_{l+1} \otimes \gamma_{\sigma_{n_0+n_1+l_2+2}} \otimes \cdots \otimes \gamma_{\sigma_{n+1}}) \\
&\quad \times [-l_2 - 1](f_{m_1+1} \otimes \cdots \otimes f_m)) \\
&= \sum \sum_{\bar{\Gamma} \in G^{l,1}_{n+1,m}} \int_{\partial_T X^l_{n+1,m}} w_{\bar{\Gamma}} \cdot \mathcal{U}_{\bar{\Gamma}}(\gamma_1 \otimes \cdots \otimes \gamma_{n+1})(f_1 \otimes \cdots \otimes f_m).
\end{aligned} \tag{37}$$

2. We consider the following situation:





Let Γ_0 be an $\emptyset$-admissible graph of type $(n_0, 2, 2n_0)$ and Γ_1 be an l-admissible graph of type $(n_1, m_1, 2n_1 + m_1 - 2 - l)$.

We denote by $S(\Gamma_0, \Gamma_1, 2)$ the set of unnumbered l-admissible graphs obtained from the two graphs Γ_0 and Γ_1 by removing the vertex $2 \in V^2_{\bar{\Gamma}}$ and connecting the edges in $\mathrm{St}'(2)$ to the vertices of Γ_1. The orderings are inherited from those of Γ.

We have the following equality:

$$\delta_1 \mathcal{U}^0_{\Gamma_0}\left(\bigotimes_{v\in\Gamma_0}\eta_v\right)\left(f_1\otimes\mathcal{U}^l_{\Gamma}\left(\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\right)(f_2\otimes\cdots\otimes f_m)\right) = \sum_{\bar{\Gamma}\in S(\Gamma_0,\Gamma_1,2)}\mathcal{U}^l_{\bar{\Gamma}}(\gamma_1\otimes\cdots\otimes\gamma_{n+1})[-l_1-1](f_1\otimes\cdots\otimes f_m), \tag{38}$$

where we put as follows:

$$\begin{aligned}\delta_1' &\equiv (-l-1)\left(|f_1|_1 + \sum_{v\in V^1_{\Gamma_0}}|\eta_v|\right) + |f_1|\cdot\sum_{v\in V^1_{\Gamma_1}}|\eta_v| - |f_1|_2\cdot\sum_{v\in V^1_{\Gamma_1}}|\eta_v|_2\\ &\equiv |f_1|\left(\sum_{\Gamma_1}|\eta_v| - 1 - l\right) - m_1 - l + (l+1)\sum_{\Gamma_0}|\eta_v| \pmod 2.\end{aligned} \tag{39}$$

Again the numbering of $\bar{\Gamma}$ is given as in the previous cases.

We consider the following coordinate transformation:

$$\begin{aligned}&(z_1,\ldots,z_{n+1};p_1,\ldots,p_m)\\ &\quad\simeq (z_{\sigma_{l+1}},\ldots,z_{\sigma_{n_0+l+1}};p_1,z_1,\ldots,z_{l+1},z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};\\ &\qquad p_2,\ldots,p_m)\,(-1)^l\\ &\quad\simeq (\tau,z_{\sigma_{l+1}},\ldots,z_{\sigma_{n_0+l+1}};p_1)\,(z_1,\ldots,z_{l+1},z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};\\ &\qquad p_2,\ldots,p_m)\,(-1)^l\\ &\quad\simeq (\text{inner normal},z_{\sigma_{l+1}},\ldots,z_{\sigma_{n_0+l+1}};p_1,\bar{p}_2)\,(z_1,\ldots,z_{l+1},\\ &\qquad z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};p_2,\ldots,p_m)\,(-1)^l.\end{aligned}$$

Then we have the following equality:

$$\begin{aligned}&(-1)^l\delta\,\mathrm{sgn}(\sigma,(\gamma_p))W_{\Gamma_0}W_{\Gamma_1}\mathcal{U}_{\Gamma_0}\left(\bigotimes_{v\in V^1_{\Gamma_0}}\eta_v\right)\\ &\quad\times\left(f_1\otimes\mathcal{U}^l_{\Gamma}\left(\bigotimes_{v\in V^1_{\Gamma_1}}\eta_v\right)[-l-1](f_2\otimes\cdots\otimes f_m)\right)\\ &\quad=\sum\mathrm{sgn}(\sigma,(\gamma_p))\int_{\partial_{\Gamma_0\Rightarrow\Gamma_i}X^l_{n+1,m}} w_{\bar{\Gamma}}\cdot\mathcal{U}_{\bar{\Gamma}}(\gamma_1\otimes\cdots\otimes\gamma_{n+1})(f_1\otimes\cdots\otimes f_m).\end{aligned} \tag{40}$$

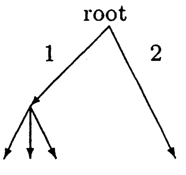
As in the previous cases, we obtain the following formula:

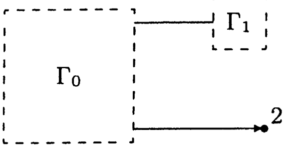
$$\begin{aligned}
&\frac{1}{n_0!(n_1-l-1)!}\sum_{\sigma\in\Sigma_{n-l}}\delta_1\,\mathrm{sgn}(\sigma,(\gamma_p))\cdot\mathcal{U}_{n_0,2}(\gamma_{\sigma_{l+2}}\otimes\cdots\otimes\gamma_{\sigma_{l+1+n_0}})\\
&\qquad\times(f_1\otimes\mathcal{U}^{l}_{n_1,m-1}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}\otimes\gamma_{\sigma_{l+2+n_0}}\otimes\cdots\otimes\gamma_{\sigma_{n+1}})(f_2\otimes\cdots\otimes f_m))\\
&\quad=\sum\sum_{\bar\Gamma\in G^{l,1}_{n+1,m}}\int_T w_{\bar\Gamma}\cdot\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{n+1}),
\end{aligned}\tag{41}$$

where we put as follows:

$$\begin{aligned}
\delta_1' \equiv{}& (l+1)|\mathcal{U}_{n_0,2}(\gamma_{\sigma_{l+2}}\otimes\cdots\otimes\gamma_{\sigma_{l+1+n_0}})[-1]|+|f_1|\cdot|\mathcal{U}_{n_1,m-1}(\gamma_1\otimes\cdots\\
&\otimes\gamma_{l+1}\otimes\gamma_{\sigma_{l+2+n_0}}\otimes\cdots\otimes\gamma_{\sigma_{n+1}})[-1-1]|+m_1 \pmod 2.
\end{aligned}\tag{42}$$

3. We consider the following situation:





We consider the following coordinate transformations:

$$\begin{aligned}
&(z_1,\ldots,z_{n+1},p_1,\ldots,p_m)\\
&\quad\simeq(z_{\sigma_{l+2}},\ldots,z_{\sigma_{n_0+l+1}};\,p_m z_1,\ldots,z_{l+1},z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};\\
&\qquad p_1,\ldots,p_{m-1})\,(-1)^{m-1+l}\\
&\quad\simeq(\tau,z_{\sigma_{l+2}},\ldots,z_{\sigma_{n_0+l+1}};\,p_m)\,(z_1,\ldots,z_{l+1},z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};\\
&\qquad p_1,\ldots,p_{m-1})\,(-1)^{m-1+l}\\
&\quad\simeq(\text{inner normal},z_{\sigma_{l+2}},\ldots,z_{\sigma_{n_0+l+1}};\,\bar p_1,p_m)(z_1,\ldots,z_{l+1},\\
&\qquad z_{\sigma_{n_0+l+2}},\ldots,z_{\sigma_{n+1}};\,p_1,\ldots,p_{m-1})\,(-1)^{m-2+l}.
\end{aligned}$$

We obtain the following formula in this case:

$$\begin{aligned}
&\frac{1}{n_0!(n_1-l-1)!}\sum_{\sigma\in\Sigma_{n-l}}\delta\,\mathrm{sgn}(\sigma,(\gamma_p))\cdot\mathcal{U}_{n_0,2}(\gamma_{\sigma_{l+2}}\otimes\cdots\\
&\quad\otimes\gamma_{\sigma_{l+1+n_0}})(\mathcal{U}^{l}_{n_1,m-1}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}\otimes\gamma_{\sigma_{l+2+n_0}}\otimes\cdots\\
&\quad\otimes\gamma_{\sigma_{n+1}})(f_1\otimes\cdots\otimes f_{m-1})\otimes f_m)\\
&=\sum\sum_{\bar\Gamma\in G^{l,1}_{n+1,m}}\int_T w_{\bar\Gamma}\cdot\mathcal{U}_{\bar\Gamma}(\gamma_1\otimes\cdots\otimes\gamma_{n+1})(f_1\otimes\cdots\otimes f_m),
\end{aligned}\tag{43}$$

where we put as follows:

$$\delta' \equiv (l+1)|\mathcal{U}_{\Gamma_{n_0},2}(\gamma_{\sigma(l+2)} \otimes \cdots \otimes \gamma_{\sigma(l+1+n_0)})[-1]| + m \pmod 2. \tag{44}$$

The general case: We give only the result on the difference of the signature. First we see the following lemma. We put $\bar{n}_i = \sum_{p=1}^{i} n_p$.

Lemma 3.3. *We consider the following transition*:

$$\begin{aligned}&\bigotimes_{v\in V^1_{\Gamma_0}} \eta_v \otimes \bigotimes_{q=1}^{j_1-1} f_q \otimes \prod_{t=1}^{k}\left(\left(\bigotimes_{v\in V^1_{\Gamma_t}} \eta_v \otimes \bigotimes_{q=j_t}^{j_t+m_t-1} f_q\right)\right.\\ &\left.\quad\times\left[-\sum_{v\in V^1_{\Gamma_t}} |\eta_v| - m_t + 1\right] \otimes \bigotimes_{q=j_t+m_t}^{j_{t+1}-1} f_q\right) \mapsto \bigotimes_{v\in V^1_{\bar{\Gamma}}} \eta_v \otimes \bigotimes_{i=1}^{m} f_i.\end{aligned} \tag{45}$$

Then the dg-symmetric signature along the transition is as follows:

$$\begin{aligned}&\sum_{q=1}^{k}\left(\sum_{t=1}^{n_q}|\gamma_{\sigma(\bar{n}_{q-1}+t)}|\right)\cdot\sum_{p=1}^{j_q-1}|f_p| - \sum_{q=1}^{k}\left(\sum_{t=1}^{n_q}|\gamma_{\sigma(\bar{n}_{q-1}+1)}|_2\right)\cdot\sum_{p=1}^{j_q-1}|f_p|_2 + t(\sigma)'\\ &\quad+\sum_{q=1}^{k}\left(\sum_{t=1}^{n_q}|\gamma_{\sigma(\bar{n}_{q-1})}|_2 - m_q + 1\right)\cdot\left(\sum_{s=1}^{\bar{n}_{q-1}}|\gamma_{\sigma(s)}|_1 + \sum_{p=1}^{j_q-1}|f_p|_1\right)\\ &\equiv \sum_{q=1}^{k}\left(\sum_{t=1}^{n_q}|\gamma_{\sigma(\bar{n}_{q-1}+1)}|_1\right)\cdot\sum_{p=1}^{j_q-1}|f_p| + \sum_{q=1}^{k}(-m_p+1)\left(\sum_{p=1}^{j_q-1}|f_p|_1\right)\\ &\quad+\sum_{q=1}^{k}(l_q+1)\left(\sum_{s=1}^{\bar{n}_{q-1}}|\gamma_{\sigma(s)}|_1\right) + t(\sigma)'.\end{aligned} \tag{46}$$

On the other hand, we obtain the following lemma.

Lemma 3.4. *The difference of the signature between* $\prod W_{\Gamma_p}$ *and* $\int_\partial w_{\bar{\Gamma}}$ *is*

$$\begin{aligned}&\sum_{i=1}^{k}\sum_{j=0}^{i-1}(m_j-1)(l_i+1) + \sum(m_i-1)(j_i-1)\\ &\quad+k+\sum_{i=1}^{k}(m_i-1)\sum_{j=0}^{i-1}(m_j-1) + \operatorname{sgn}(\sigma, (|\gamma_p|_2))'.\end{aligned}$$

Proof. It can be shown by the comparison of the orientations as in the previous cases. □

Hence the total difference of the signature is the following in $\mathbb{Z}/2\mathbb{Z}$:

$$\sum_{q=1}^{k}\sum_{p=1}^{j_q-1}|f_p|\left(\sum_{t=1}^{n_q}|\gamma_{\sigma(\bar{n}_{q-1}+t)}|_1-m_q+1\right)+\sum_{q=1}^{k}(l_q+1)\left(\sum_{s=1}^{\bar{n}_{q-1}}|\gamma_{\sigma(s)}|+l_s+1\right)$$
$$+\sum_{i=1}^{k}(m_i-1)\sum_{j=0}^{i-1}(m_j-1)+k+\mathrm{sgn}(\sigma,\gamma_p)' \pmod 2. \tag{47}$$

It can be rewritten as follows:

$$\sum_{q=1}^{k}\sum_{p=1}^{j_q-1}|f_p|\left|\mathcal{U}_{\Gamma_q}\left(\bigotimes_{v\in\Gamma_q}\eta_v\right)[-l_q-1]\right|+\sum_{q=1}^{k}(l_q+1)\sum_{t=0}^{q-1}\left|\mathcal{U}_{\Gamma_t}\left(\bigotimes_{v\in\Gamma_t}\eta_v\right)[-l_t-1]\right|$$
$$+\sum_{i=1}^{k}(m_i-1)\sum_{j=0}^{i-1}(m_j-1)+k+\mathrm{sgn}(\sigma,\gamma_p)' \pmod 2. \tag{48}$$

3.3. Formality theorem

In the case $l=0$, the contribution of types I and II appears, and we obtain the following equality:

$$\begin{aligned}
0 &= \sum_{i<j}(-1)^{|\gamma_i[-1]|+\sum_{k\le i-1}|\gamma_k|}\mathrm{sgn}(\kappa_{ij},(\gamma_p))\\
&\quad\times\mathcal{U}_n^0(\gamma_1\otimes\cdots\otimes(\gamma_i[-1]\bullet\gamma_j[-1])[1]\otimes\cdots\otimes\gamma_{n+1})\\
&\quad+\sum_{i<j}(-1)^{|\gamma_j[-1]|+\sum_{k\le i-1}|\gamma_k|}\,\mathrm{sgn}(\kappa_{ji},(\gamma_p))\\
&\quad\times\mathcal{U}_n^0(\gamma_1\otimes\cdots\otimes(\gamma_j[-1]\bullet\gamma_i[-1])[1]\otimes\cdots\otimes\gamma_{n+1})\\
&\quad-\sum_{\substack{n_0+n_1=n+1\\ m_0+m_1=m+1}}\frac{1}{(n_0-1)!n_1!}\\
&\quad\times\sum_{\sigma\in\Sigma_n}\mathrm{sgn}(\sigma,(\gamma_p))(-1)^{(m_0+1)(m_1+1)+\mathcal{U}^0_{n_0,m_0}(\gamma_1\otimes\gamma_{\sigma_2}\otimes\cdots\otimes\gamma_{\sigma_{n_0}})[-1]|}\\
&\quad\times\mathcal{U}^0_{n_0,m_0}(\gamma_1\otimes\gamma_{\sigma_2}\otimes\cdots\otimes\gamma_{\sigma_{n_0}})[-1]\circ_i\mathcal{U}^0_{n_1,m_1}(\gamma_{\sigma_{n_0+1}}\otimes\cdots\otimes\gamma_{\sigma_{n+1}})[-1]\\
&\quad-\sum_{\substack{n_0+n_1=n+1\\ m_0+m_1=m+1}}\frac{1}{(n_0)!(n_1-1)!}\\
&\quad\times\sum_{\sigma\in\Sigma_n}\mathrm{sgn}(\sigma,(\gamma_p))(-1)^{(m_1+1)(m_0+1)+|\mathcal{U}^0_{n_0,m_0}(\gamma_{\sigma_2}\otimes\cdots\otimes\gamma_{\sigma_{n_0+1}})|}\\
&\quad\times\mathcal{U}^0_{n_0,m_0}(\gamma_{\sigma_2}\otimes\cdots\otimes\gamma_{\sigma_{n_0+1}})\circ_i\mathcal{U}^0_{n_1,m_1}(\gamma_1\otimes\gamma_{\sigma_{n_0}}\otimes\cdots\otimes\gamma_{\sigma_n})[-1].
\end{aligned}\tag{49}$$

It can be rewritten as follows:

$$\begin{aligned}
0 ={}& \sum_{i<j}(-1)^{|\gamma_i[-1]|+\sum_{k\le i-1}|\gamma_k|}\,\mathrm{sgn}(\kappa_{ij},(\gamma_p))\mathcal{U}^0_n(\gamma_1\otimes\cdots\otimes(\gamma_i\bullet\gamma_j)\otimes\cdots\otimes\gamma_{n+1})\\
&+\sum_{i<j}(-1)^{|\gamma_i[-1]|+\sum_{k\le i-1}|\gamma_k|}\mathrm{sgn}(\kappa_{ji},(\gamma_p))\mathcal{U}^0_n(\gamma_1\otimes\cdots\otimes(\gamma_j\bullet\gamma_i)\otimes\cdots\otimes\gamma_{n+1})\\
&-\sum_{\substack{n_0+n_1=n+1\\ m_0+m_1=m+1}}\frac{1}{(n_0)!(n_1)!}\\
&\times\sum_{\sigma\in\Sigma_{n+1}}\mathrm{sgn}(\sigma,(\gamma_p))(-1)^{(m_0+1)(m_1+1)+|\mathcal{U}^0_{n_0,m_0}(\gamma_{\sigma_1}\otimes\cdots\otimes\gamma_{\sigma_{n_0}})|}\\
&\times\mathcal{U}^0_{n_0,m_0}(\gamma_{\sigma_1}\otimes\cdots\otimes\gamma_{\sigma_{n_0}})\circ_i\mathcal{U}^0_{n_1,m_1}(\gamma_{\sigma_{n_0+1}}\otimes\cdots\otimes\gamma_{\sigma_{n+1}}).
\end{aligned}\tag{50}$$

The equality shows the L_∞-property of the morphism $\mathcal{U}=\mathcal{U}^0$ as is seen in Section 2.5.

3.4. *Cup products*

Take an element $\alpha\in T_{\mathrm{poly}}[1]$ such that the shift $\alpha[-1]\in T_{\mathrm{poly}}$ is a solution of the Maurer–Cartan equation for $T_{\mathrm{poly}}(\mathbb{R}^d)$, which gives a solution $\bar{\alpha}[-1]$ of the Maurer–Cartan equation for the $D_{\mathrm{poly}}(\mathbb{R}^d)$, we put as follows:

$$\bar{\alpha}=\sum_{n\ge1}\frac{t^n}{n!}\mathcal{U}_n(\overbrace{\alpha\cdots\alpha}^{n}).$$

We denote the multiplication of the $C^\infty(\mathbb{R}^d)$ by μ. We put $\bar{\mu}=\mu+\bar{\alpha}$. We have the decomposition $\bar{\mu}=\sum\bar{\mu}_n$.

The solution α gives the formal deformation of the differential graded Lie algebra $T_{\mathrm{poly}}[[t]]$ by putting the differential as $dx=t[\alpha,x]$. The complex $(T_{\mathrm{poly}}[-1][[t]],d)$ is the differential graded Lie algebra with the cup product. The solution $\bar{\alpha}$ gives the A_∞-deformation of the associative $D_{\mathrm{poly}}[-1][[t]]$ by putting the differentials $d+\bar{\alpha}$, i.e., $dx=[\bar{\mu},x][1]$. We denote them by $(T_{\mathrm{poly}})_\alpha$, $(D_{\mathrm{poly}})_{\bar{\alpha}}$, respectively.

We define the morphism $\mathcal{U}^{l,\alpha}_{n,m}:\otimes^{l+1}T_{\mathrm{poly}}[1]\to D_{\mathrm{poly}}[1+l]$ as follows:

$$\mathcal{U}^{l,\alpha}_{n,m}(\gamma_1\otimes\cdots\otimes\gamma_{l+1})=\mathcal{U}^l_{n,m}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}\otimes\alpha\otimes\cdots\otimes\alpha).$$

We define the morphism $T_\alpha\mathcal{U}:T_{\mathrm{poly}}[1]\to D_{\mathrm{poly}}[1][[t]]$ as follows:

$$T_\alpha\mathcal{U}(\gamma)=\sum_{n,m}\frac{t^{n-1}}{(n-1)!}\mathcal{U}^{0,\alpha}_{n,m}(\gamma\cdot\overbrace{\alpha\cdots\alpha}^{n-1}).$$

Moreover, we define the morphism $T\mathcal{U}^{l,\alpha}:\otimes^{l+1}T_{\mathrm{poly}}[1]\to D_{\mathrm{poly}}[1+l][[t]]$ as follows:

$$T\mathcal{U}^{l,\alpha}(\gamma_1\otimes\cdots\otimes\gamma_{l+1})=\sum_{n,m}\frac{t^{n-l-1}}{(n-l-1)!}\mathcal{U}^{l,\alpha}_{n,m}(\gamma_1\otimes\cdots\otimes\gamma_{l+1}).$$

We have $T_\alpha\mathcal{U}=T\mathcal{U}^{0,\alpha}$. We put $x_i=\gamma_i[-2]$.

The contribution of type I to the integral $\int_{X^l_{n+1,m}} d\bar{w}_\Gamma$ is summed as follows:

$$(n-l)(-1)^{\sum_{p<i}|\gamma_p|+1}\mathcal{U}^{l,\alpha}_{n,m}(\gamma_1\otimes\cdots\otimes(\alpha[-1]\bullet\gamma_i[-1])[1]\otimes\cdots\otimes\gamma_{l+1})[1]$$
$$+(n-l)(-1)^{\sum_{p<i}|\gamma_p|+|\gamma_i|-1}\mathcal{U}^{l,\alpha}_{n,m}(\gamma_1\otimes\cdots\otimes(\gamma_i[-1]\bullet\alpha[-1])[1]$$
$$\otimes\cdots\otimes\gamma_{l+1})[1]. \tag{51}$$

We have the following equality:

$$\Bigl(\sum(-1)^{\sum_{p<i}|\gamma_p|}\gamma_1\otimes\cdots\otimes(-\alpha[-1]\bullet\gamma_i[-1]+(-1)^{|\gamma_i|-1}\gamma_i[-1]$$
$$\bullet\alpha[-1])[1]\otimes\cdots\otimes\gamma_{l+1}\Bigr)[1]$$
$$=\Bigl(\sum(-1)^{\sum_{p<i}|\gamma_p|}\gamma_1\otimes\cdots\otimes(-[\alpha[-1],\gamma_i[-1]][1])\otimes\cdots\otimes\gamma_{l+1}\Bigr)[1]$$
$$=t^{-1}d(\gamma_1\otimes\cdots\otimes\gamma_{l+1}). \tag{52}$$

Thus we obtain the following as the contribution of type I:

$$(n-l)\mathcal{U}^{l,\alpha}_{n,m}t^{-1}(d(x_1\otimes\cdots\otimes x_{l+1})).$$

The contribution of type III is the following:

$$\sum(-1)^i\mathcal{U}^{l-1,\alpha}_{n,m}(\gamma_1\otimes\cdots\otimes(\gamma_i[-2]\wedge\gamma_{i+1}[-2])[2]\otimes\cdots\otimes\gamma_{l+1})[1].$$

We rewrite it as follows:

$$-\mathcal{U}^{l-1}_{n,m}(\wedge(x_1\otimes\cdots\otimes x_{l+1})).$$

The contribution of type II can be written as follows:

$$\sum\frac{(n-l)!}{(n_0-l-1)!n_1!}(-1)^{(m_1+1)(m_0+1)+(l+1)+|(\mathcal{U}^{l,\alpha}_{n_0,m_0}(\gamma_1\otimes\cdots\otimes\gamma_{l+1})[-1-l])|}$$
$$\times(\mathcal{U}^{l,\alpha}_{n_0,m_0}(\gamma_1\otimes\cdots\otimes\gamma_{l+1})[-1-l]\circ_i\tilde{\mu}_{n_1}[-1])[2+l]. \tag{53}$$

The contribution of type IV can be written as follows:

$$\sum_{\substack{k\\ \sum(l_p+1)=l+1}}\frac{(n-l)!}{n_0!\prod_{p=1}^k(n_p-l_p-1)!}\delta\cdot\pi(F(\bar{\mu}_{n_0})$$
$$\circ\square(\mathcal{U}^{l_1,\alpha}_{n_1,m_1}(\gamma_1\otimes\cdots\otimes\gamma_{l_1+1})[-l_1-1]\otimes\cdots$$
$$\otimes\mathcal{U}^{l_k,\alpha}_{n_k,m_k}(\gamma_{l-l_k}\otimes\cdots\otimes\gamma_{l+1})[-l_k-1]))[l+2], \tag{54}$$

where we put as follows:

$$\delta'=\sum_{q=1}^k(l_q+1)\left(\sum_{s=0}^{q-1}\left|\mathcal{U}_{\Gamma_s}\left(\bigotimes_{v\in V^1_{\Gamma_s}}|\eta_v|\right)[-l_s+1]\right|\right)$$
$$+\sum_{i=1}^k(m_i-1)\sum_{j=0}^{i-1}(m_j-1)+k\pmod 2.$$

Thus we obtain the following equality:

$$\begin{aligned}
&T\mathcal{U}^{l,\alpha}(d(x_1\otimes\cdots\otimes x_{l+1}))-T\mathcal{U}^{l-1,\alpha}(\wedge(x_1\otimes\cdots\otimes x_{l+1}))\\
&\quad-\mathrm{ad}(\bar{\mu})(T\mathcal{U}^{l,\alpha}(x_1\otimes\cdots\otimes x_{l+1}))\\
&\quad+\sum_{\substack{\sum_{i=1}^{k}(l_i+1)=l+1\\ k\geq 2}}\pi(F(\bar{\mu})\bar{\circ}\Box(T\mathcal{U}^{l_1,\alpha}(x_1\otimes\cdots\otimes x_{l_1+1})\otimes\cdots\\
&\quad\otimes T\mathcal{U}^{l-l_k+1,\alpha}(x_1\otimes\cdots\otimes x_{l+1})))=0. \qquad (55)
\end{aligned}$$

It is the A_∞-property of the morphism $(T\mathcal{U}^{l,\alpha}|l=0,1,2,\ldots)$.

In particular, we consider the case $A=C^\infty(\mathbb{R}^n)$. In this case, we have that $\bar{\mu}=\bar{\mu}_2$. We obtain the following as the equality in the case $l=1$:

$$\begin{aligned}
&T\mathcal{U}^{1,\alpha}(d(x_1\otimes x_2))-T\mathcal{U}^{0,\alpha}(x_1\wedge x_2)-\mathrm{ad}(\bar{\mu})(T\mathcal{U}^{1,\alpha}(x_1\otimes x_2))\\
&\qquad+\bar{\mu}(T\mathcal{U}^{0,\alpha}(x_1)\otimes T\mathcal{U}^{0,\alpha}(x_2))=0. \qquad (56)
\end{aligned}$$

It implies that the compatibility of the products in the cohomology level.

4. The RKV conjecture

4.1. The convolution product and its deformations

There is the convolution structure on the space of distributions $\mathcal{D}'(\mathfrak{g})$ from the structure of vector space, i.e.,

$$*_{\mathfrak{g}}:\mathcal{D}'(\mathfrak{g})\times\mathcal{D}'(\mathfrak{g})\to\mathcal{D}'(\mathfrak{g}),\qquad u*_{\mathfrak{g}}v(\phi)=p_1^*(u)\boxtimes p_2^*(v)(m_0^*(\phi)).$$

We denote the projection $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ to the ith component by p_i. We denote the summation $(g,h)\mapsto g+h$ by m_0. We put $\mathcal{D}'(\mathfrak{g})[[t]]:=\mathcal{D}'(\mathfrak{g})\otimes\mathbb{C}[[t]]$.

We construct two deformations of the product structure as follows:

1. The first is the standard deformation by the structure of Lie group. Let $U\subset\mathfrak{g}$ be a neighborhood of 0 such that the map $\exp|_U:U\to U'=\exp(U)\subset G$ is a diffeomorphism. We denote the space of vector fields on the set U by $\mathfrak{X}(U)$. We have the two inclusions of the $\mathfrak{g}$ into the $\mathfrak{X}(U)$: any $X\in\mathfrak{g}$ gives the vector field X with the constant coefficients, i.e., $Xf(x)=(d/dt)f(x+tX)$. On the other hand, any $X\in\mathfrak{g}$ induces the left invariant vector field $\tilde{X}$ on G, which gives the vector field $(\exp_*)^{-1}\tilde{X}$ on U.

   We put $X_t(x)=(\exp_*)^{-1}\tilde{X}(tx)\in\mathfrak{X}(U)$ for any $t\in[0,1]$. Obviously, it holds that $X_0=X$ and that $X_1=\exp_*^{-1}\tilde{X}$. We have the relation $[X_t,Y_t]=t[X,Y]_t$.

   1.1. We obtain the pseudo-product structure $(X,Y)\to m_t(X,Y)$: we can take an open set $V\subset U$ for any $t\in[0,1]$, there is the analytic map $m_t:V\times V\to U$ given by the following:

   $$m_t(X,Y)=\mathrm{Exp}(X_t)\circ\mathrm{Exp}(Y_t)(0)=(\mathrm{Exp}(X_t))\exp(Y),$$

where $\mathrm{Exp}(X_t)$ is the map obtained by the vector field X_t. Obviously, it holds that $m_0(X, Y) = X + Y$ and that $m_1(X, Y) = \exp^{-1}(\exp(X)\exp(Y))$. Thus we obtain the family to connect the summation of the Lie algebra and the multiplication of the Lie group.

1.2. For any $t \in [0, 1]$, we obtain the deformation $*_t : \mathcal{D}'(K_1, V) \times \mathcal{D}'(K_2, V) \to \mathcal{D}'(K_1 + K_2, V)$ of the convolution product

$$u *_t v(\phi) = p_1^*(u) \boxtimes p_2^*(v)(m_t^*(\phi)).$$

The $*_0$ is the convolution for $\mathfrak{g}$ and the $*_1$ is the convolution for G.

For any $u \in \mathcal{D}'(K_1, V)$, $v \in \mathcal{D}'(K_2, V)$ and a test function ϕ, we have the C^∞-function $(u *_t v)(\phi)$ of the variable t. We have the (not necessarily convergent) Taylor series at $t = 0$ of the $(u *_t v)(\phi)$ which we denote by $\sum t^n Q_n(\phi)$. The Q_n is a differential operator with at most $2n$th order of analytic coefficients. Thus the $*_t$ induces the formal deformation which we denote also by $*_t$.

2. The second deformation is essentially due to Kontsevich:

2.1. Let $\alpha \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ be the canonical tensor which gives the bracket of Lie algebra. We put

$$\mathcal{U}_t(\alpha) = \sum_{n\geq 0} \frac{t^n}{n!} \mathcal{U}_n(\overbrace{\alpha \wedge \cdots \wedge \alpha}^{n}) \in \mathrm{Sym}(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}).$$

Here we regard the variable t as a formal parameter.

The $\mathcal{U}_n(\overbrace{\alpha \wedge \cdots \wedge \alpha}^{n})$ is the differential operator with at most $2n$-order of polynomial coefficients.

3. We obtain the formal deformation $\star$ with the formal parameter t of the convolution product $\mathcal{D}'(\mathfrak{g})[[t]] \otimes \mathcal{D}'(\mathfrak{g})[[t]] \to \mathcal{D}'(\mathfrak{g})[[t]]$:

$$u \star v(\phi) = p_1^*(u) \boxtimes p_2^*(v)(\mathcal{U}_t(\alpha) m_0^*(\phi)).$$

Due to the definition, the $u \star v(\phi)$ is the formal power series $\sum P_n(\phi) t^n$. The P_n are differential operators with at most $2n$th order of analytic coefficients.

4.2. Proof of the conjecture

Kontsevich obtained the formal power series $j_{1,t}, j_{2,t} \in \mathbb{C}[\mathfrak{g}^*][[t]]$ (see [10]) of the form

$$j_{i,t} = \exp\left(\sum_{k\geq 1} c_{2k}^{(i)} t^{2k} \mathrm{Tr}((\mathrm{ad}\,\alpha)^{2k})\right).$$

Proposition 4.1. *We consider the following map* $(\mathcal{D}'_0(\mathfrak{g})[[t]], *_t) \to (\mathcal{D}'_0(\mathfrak{g})[[t]], \star)$, $u \mapsto j_{2,t} u$. *The two formal deformations of product structures are equivalent under the morphism.*

Proof. We denote the Taylor series with respect to the variable t of the $p_1^*(j_2 u) \boxtimes p_2^*(j_2 v)\, \mathcal{U}_t m_0^*(\phi)$ (resp. $p_1^*(u) \boxtimes p_2^* m_t^*(j_2 \phi)$) by $\sum P_n(\phi) t^n$ (resp. $\sum Q_n(\phi) t^n$). Also, we denote the Taylor series with respect to the variable t of the $p_1^*(j_2)p_2^*(j_2)\mathcal{U}_t m_0^*(\phi)$ (resp. $m_t^*(j_2\phi)$) by $\sum \bar{P}_n(\phi)t^n$ and $\sum \bar{Q}_n(\phi)t^n$. To compare the $P_n(\phi)$ and the $Q_n(\phi)$, we only have to compare the $\bar{P}_n(\phi)$ and the $\bar{Q}_n(\phi)$. Note that the $\bar{P}_n(\phi)$ and the $\bar{Q}_n(\phi)$ are defined for any smooth function which are not necessarily test functions.

Lemma 4.1. *For any analytic function f, we have the equality* $\bar{P}_n(f) = \bar{Q}_n(f)$.

Proof. The restriction of the $*_t$ to the space of distributions with 0-support is the product of the enveloping algebra of the deformed Lie algebra with the bracket $[X, Y]_t = t[X, Y]$. On the other hand, the restriction of the $\star$ is the product constructed by Kontsevich (see Section 1). Thus the formal power series $u\star v$ for the variable t converges for any distribution u, v with 0-support, and we know that $u\star v = u *_t v$ in this case. Since the supports of u and v are $\{0\}$, $u *_t v(f)$ and $u\star v(f)$ are defined for any analytic function, and they coincide, i.e., $u *_t v(f) = u\star v(f)$. It implies the equalities $p_1^*(u)\boxtimes p_2^*(v)\bar{P}_n(f) = p_1^*(u)\boxtimes p_2^*(v)\bar{Q}_n(f)$. Thus all of the Taylor coefficients of the $\bar{P}_n(f)$ and the $\bar{Q}_n(f)$ coincide. If f is analytic, then the $\bar{P}(f)$ and the $\bar{Q}(f)$ are analytic. Thus we are done. $\square$

Lemma 4.2. *For any test function* ϕ *and distributions* $u, v \in \mathcal{D}'(U)$, *it holds that*

$$p_1^*(\psi u) \boxtimes p_2^*(\psi v)(\bar{P}_n(\phi)) = p_1^*(\psi u) \boxtimes p_2^*(\psi v)(\bar{Q}_n(\phi)).$$

Proof. For any analytic function f defined on U, the function $\bar{Q}_n(f) - \bar{P}_n(f)$ is constantly 0 due to the previous lemma. We can take a sequence of analytic functions f_i which converges to ϕ on U with respect to any C^N-norm $\|\cdot\|_{C^N,U}$, $N = 1, 2, \dots$.

The sequence $\bar{Q}_n(f_i) - \bar{P}_n(f_i)$ converges to $\bar{Q}_n(\phi) - \bar{P}_n(\phi)$. Thus we are done. $\square$

Thus we can conclude that $P_n(\phi) = Q_n(\phi)$ for any test function ϕ, which implies the claim of the proposition.

Next we consider the following map Ψ

$$(\mathcal{D}'(\mathfrak{g})[[t]], *_0) \to (\mathcal{D}'(\mathfrak{g})[[t]], \star), \quad u \mapsto j_1 u.$$

Proposition 4.2. *The restriction of the morphism to the relative invariant part preserves the product structure.*

Proof. We have constructed the morphism $T\mathcal{U}^{1,\alpha}$ in the previous section

$$(\mathrm{Sym}\,\mathfrak{g} \otimes \mathfrak{g}^*) \otimes \mathrm{Sym}\,\mathfrak{g} \to \mathrm{Sym}\,\mathfrak{g}[[t]], \qquad \mathrm{Sym}\,\mathfrak{g} \otimes \mathrm{Sym}\,\mathfrak{g} \otimes \mathfrak{g}^* \to \mathrm{Sym}\,\mathfrak{g}[[t]],$$

There are a $\mathfrak{g}[[t]]$-valued polynomial function f on the $\mathfrak{g}\times\mathfrak{g}$ and an $\mathbb{R}[[t]]$-valued polynomial function g on the $\mathfrak{g} \times \mathfrak{g}$, so that for any element $u \otimes \beta \in \mathrm{Sym}(\mathfrak{g}) \otimes \mathfrak{g}^*$ and $v \in \mathrm{Sym}(\mathfrak{g})$, the $T\mathcal{U}^{\alpha,1}(u \otimes \beta \otimes v)$ is of the form

$$T\mathcal{U}^{\alpha,1}((u \otimes \beta) \otimes v) = (f(\beta)p_1^*u \boxtimes p_2^*v + gp_1^*u \boxtimes p_2^*\beta v)m_0^*.$$

The $T\mathcal{U}^{\alpha,1}(u \otimes (v \otimes \beta))$ has a similar form. Thus the morphism $T\mathcal{U}^{\alpha,1}$ can be naturally extended as the morphism

$$(\mathcal{D}'(\mathfrak{g}) \otimes \mathfrak{g}^*) \otimes \mathcal{D}'(\mathfrak{g}) \to \mathcal{D}'(\mathfrak{g}), \qquad \mathcal{D}'(\mathfrak{g}) \otimes (\mathcal{D}'(\mathfrak{g}) \otimes \mathfrak{g}^*) \to \mathcal{D}'(\mathfrak{g}).$$

The calculation in the previous section shows that it holds that

$$j_1(u *_{\mathfrak{g}} v) - j_1(u) \star j_1(v) = \pm T\mathcal{U}^{\alpha,1}(\mathrm{ad}(\alpha)(u) \otimes v) \pm T\mathcal{U}^{\alpha,1}(\mathrm{ad}(\alpha)(v) \otimes u).$$

Note that the Leibniz rule $e_i^*(u *_{\mathfrak{g}} v) = e_i^*(u) *_{\mathfrak{g}} v + u *_{\mathfrak{g}} (e_i^* v)$ holds for the linear function e_i^* on the $\mathfrak{g}$, i.e., the element of the $\mathfrak{g}^*$, which assures that the formalism of the proof of the Formality theorem works in this case.

Let $\{e_i\}$ be a basis of the $\mathfrak{g}$ and $\{e_j^*\}$ be a dual basis of the $\mathfrak{g}^*$. The tensor α can be written as $\alpha = \sum a_{ij}^k e_i^* \otimes e_j^* \otimes e_k$. Then the vector field $\mathrm{ad}(\alpha)$ on the $\mathfrak{g}$ can be written as $\sum a_{ij}^k e_i^* e_j^* \,\mathrm{ad}(e_k)$, where $X_i = e_i^*(X)$. We have the following easy lemma.

Lemma 4.3. *It holds that*

$$\sum a_{ij}^k (e_k \delta) *_{\mathfrak{g}} (e_j^* u) \otimes e_i^* = D_{e_i} u \otimes e_i^* + \tilde{\chi}_0 u,$$

where δ is the delta function whose support is the origin of $\mathfrak{g}$, and $\tilde{\chi}_0$ denotes the infinitesimal character of χ_0.

Proof. It follows from the following direct calculation:

$$\sum a_{ij}^k X_i (e_k \delta) *_{\mathfrak{g}} (e_j^* u) = a_{ij}^j X_i u + a_{ij}^k X_i e_j^* e_k * u = \delta \chi_0(X) u + D_X u. \qquad \square$$

The formula in the previous section shows that the products on the $\mathfrak{g}$-invariant parts (i.e., 0th cohomology groups) are compatible under the morphism $T_\alpha \mathcal{U} = j_1 \cdot$. Hence the map $\Psi_{\mathcal{I}(\mathfrak{g})}$ preserves the product structure.

Therefore the isomorphism of the formal deformations

$$(\mathcal{I}(\mathfrak{g})[[t]], *_t) \to (\mathcal{I}(\mathfrak{g})[[t]](\mathfrak{g}), *_0), \quad u \mapsto j_1^{-1} \cdot j_2 \cdot u$$

preserves the product structures, i.e., for any $u \in \mathcal{I}(K_1, \mathfrak{g})$, $v \in \mathcal{I}(K_2, \mathfrak{g})$ all of the Taylor series with respect to the variable t of the $u *_0 v(\phi)$ and the $j_1 j_2^{-1}(j_1^{-1} j_2 u) *_t (j_1^{-1} j_2 v)(\phi)$ coincide for any test function ϕ.

We have the (not formal) deformations m_0 and m_t for any $t \in [0, 1]$ because of the construction. The formal power series $j_1^{-1} \cdot j_2$ about t and on the $\mathfrak{g}$ is also convergent.

We take a cut function ψ of U. Let W be a relative compact open subset of the U, which contains the origin 0. Let ψ be a function on the U such that $\psi(x) = 1$ on the W and the support of the ψ is a compact set of the U. The germ of the u and the ψu are same. For any analytic function f, $(\psi u) *_0 (\psi v)(f)$ and $j_1 j_2^{-1}(j_1^{-1} j_2 \psi u) *_t (j_1^{-1} j_2 \psi v)(f)$ are defined due to the cut function and analytic with respect to the variable t. Thus the coincidence of the Taylor series gives the equality $(\psi u) *_0 (\psi v)(f) = j_1 j_2^{-1}(j_1^{-1} j_2 \psi u) *_t (j_1^{-1} j_2 \psi v)(f)$ for any analytic function f and for any $t \in [0, 1]$. For any test function ϕ on U, we can take a sequence of analytic functions f_i which converges to ϕ on any compact set K contained in U with respect to any C^N-norm $\|\cdot\|_{C^N,U}$, $N = 1, 2, \ldots$. Thus we can conclude that

$$(\psi u) *_0 (\psi v)(\phi) = j_1 j_2^{-1}(j_1^{-1} j_2 \psi u) *_t (j_1^{-1} j_2 \psi v)(\phi)$$

for any test function ϕ and for any $t \in [0, 1]$.

Thus the two product structures on the spaces of germs of distributions $(\mathcal{D}'_e(\mathfrak{g}), *_0)$ and the $(\mathcal{D}'_e(\mathfrak{g}), *_1)$ are compatible under the morphism of the multiplication $j_{1,t} j_{2,t}^{-1}$.

Kontsevich showed that the function obtained by the substitution $t = 1$ to the $j_{1,t}^{-1} j_{2,t}$ is $j^{1/2}$. Hence we are done.

5. The BGRT conjecture

First we give the outline of the proof. We introduce the two Lie algebras which act on the $\mathcal{B}$. They are the natural counterparts of the $T_{\mathrm{poly}}(\mathbb{R}^d)$ and the $D_{\mathrm{poly}}(\mathbb{R}^d)$. We can translate the theory in the previous sections to this situation. Hence we have the following:

1. We can deform the algebra $(\mathcal{B}, m_{\mathcal{B}})$. We denote the resulting algebra by $(\mathcal{B}, \star)$.
2. We obtain the algebra homomorphism of the $(\mathcal{B}, m_{\mathcal{B}})$ to the $(\mathcal{B}, \star)$.
3. We construct the natural algebra homomorphism of the $(\mathcal{A}, m_{\mathcal{A}})$ to the $(\mathcal{B}, \star)$.
4. There is the PBW isomorphism from the $\mathcal{B}$ to the $\mathcal{A}$.

Then we obtain the algebra homomorphism of the $(\mathcal{B}, m_{\mathcal{A}})$ to the $(\mathcal{B}, m_{\mathcal{B}})$. The combinatorics of the construction is the same as that of Kontsevich [10], and hence the resulting morphism is Duflo–Kirillov morphism.

5.1. Two differential graded Lie algebras

We introduce the following two definitions.

Definition 5.1. We call the set of univalent vertices of Chinese character Γ by legs of Γ and denote the set of legs by $\mathrm{Leg}(\Gamma)$.

Definition 5.2. An m-Chinese graph is a Chinese character with the decomposition

$$\mathrm{Leg}(\Gamma) = L_\emptyset \cup \coprod_{i=1}^{m} L_i(\Gamma).$$

We denote the set of m-Chinese graph by CG_m. We put $\mathrm{CG} := \prod_m \mathrm{CG}_m$. We put $\mathrm{CO}_m := \mathrm{span}(\mathrm{CG}_m)$ and call it the group of the m-Chinese operators. We put $\mathrm{CO} := \coprod_m \mathrm{CO}_m$ and call it the groups of Chinese operators. The gradation of CO is given as $\mathrm{CO}^m = \mathrm{CO}_{m+1}$.

There is the morphism of the group CO to the differential graded Lie algebra $G(\mathcal{B}) = \oplus \mathrm{Hom}(\otimes^{m+1}\mathcal{B}, \mathcal{B})$, whose image DCO is the differential graded Lie subalgebra; i.e., to a Chinese graph Φ, we associate the element $\hat{\Phi} \in \mathrm{Hom}(\otimes^{m+1}\mathcal{B}, \mathcal{B})$. $\hat{\Phi}(\gamma_0 \otimes \cdots \otimes \gamma_m)$ is the sum of the Chinese diagrams which is obtained by connecting the legs in the $\mathrm{Leg}_k(\Phi)$

and the legs of γ_k along injective morphisms $\mathrm{Leg}_k(\Phi) \to \mathrm{Leg}(\gamma_k)$ for $k = 1, \dots, m$. The vector space DCO has the gradation induced from that on the $G(\mathcal{B})$.

Proposition 5.1. *The DCO is a differential graded Lie subalgebra of the* $G(\mathcal{B})$.

Proof. It is obvious because the composition of any elements $\Phi_1, \Phi_2 \in$ DCO belongs to the DCO. □

Definition 5.3. A special m-Chinese graph Γ is an m-Chinese graph such that $\# L_i = 1$. We denote the unique element of $L_i(\Gamma)$ by $e_i(\Gamma)$ or e_i. We denote the set of special m-Chinese graphs by the SCG_m. We put $\mathrm{SCO}_m := \mathrm{span}(\mathrm{SCG}_m)$ which we call the group of special m-Chinese operators.

We have the morphism of the group SCO^m to the group $\mathrm{Hom}(\Lambda^{m+1}\mathcal{B}, \mathcal{B})$, i.e., the composition of the following morphisms:

$$\mathrm{SCO} \to \mathrm{CO} \to \mathrm{Hom}\left(\otimes^{m+1}\mathcal{B}, \mathcal{B}\right) \to \mathrm{Hom}(\Lambda^{m+1}\mathcal{B}, \mathcal{B}).$$

We denote the image by TCO^m and we put $\mathrm{TCO} = \oplus \mathrm{TCO}^m$.

We define the following operation on the TCO. For two elements $\boldsymbol{\xi} \in \mathrm{TCO}^k$ and $\boldsymbol{\eta} \in \mathrm{TCO}^l$, the element $\xi \bullet \eta \in \mathrm{TCO}^{k+l}$ is defined as follows: we consider the sum of the special Chinese graphs $\boldsymbol{\zeta}$ which is obtained by connecting the edge $e_i(\boldsymbol{\xi})$ to an edge in the $L_\emptyset(\boldsymbol{\eta})$, which we denote by $\boldsymbol{\xi} \bullet_i \boldsymbol{\eta}$, where the numbering of $\boldsymbol{\zeta}$ is given as follows:

$$e_p(\boldsymbol{\zeta}) = e_p(\boldsymbol{\xi}) \quad (p \le i-1), \qquad e_p(\boldsymbol{\zeta}) = e_{p+1}(\boldsymbol{\xi}) \quad (i \le p \le k-1),$$

$$e_p(\boldsymbol{\zeta}) = e_{p-k}(\boldsymbol{\eta}) \quad (k \le p).$$

We put $\boldsymbol{\xi} \bullet \boldsymbol{\eta} := \sum_{i=0}^{k} (-1)^{k+i} \boldsymbol{\xi} \bullet_i \boldsymbol{\eta}$. The following lemma is shown by the same formalism as that in Section 2.2.

Lemma 5.1. *It holds that* $[\boldsymbol{\xi}, \boldsymbol{\eta}] = \boldsymbol{\xi} \bullet \boldsymbol{\eta} - (-1)^{kl} \boldsymbol{\eta} \bullet \boldsymbol{\xi}$.

Using this lemma, we obtain the following proposition easily.

Proposition 5.2. *The TCO is a differential graded Lie subalgebra of the* $N(\mathcal{B})$.

5.2. Parallel construction of $\mathcal{U}_n^l$

We construct the L_∞-morphism $\mathcal{U}$ of the TCO[1] to DCO[1]. Let Γ be an l-admissible graph of type (n, m, u) and $\gamma_1, \dots, \gamma_n$ be n elements of TCO[1] such that $|\gamma_p| = \#\mathrm{St}(p) - 2$.

For any vertex i of the first type in Γ, we denote by $\mathrm{St}(i)$ the set of the edges starting at i, and denote by $\mathrm{St}'(i)$ the set of the edges ending at i. Any bijection $I_i : \mathrm{St}(i) \to \{e_0, \dots, e_k\}$ and any injection $J_i : \mathrm{St}'(i) \to L_\emptyset(\gamma_i)$ determine the Chinese operator: for any edge $i \to j$,

I_i gives the leg of γ_i and J_j gives the leg of γ_j if j is of the first type. By connecting them for each edges, we obtain the Chinese graph which we denote by $\Gamma(I_i, J_j, \{\gamma_p\})$. Then we obtain the following Chinese operator:

$$\mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n) = \sum_{(I_i, J_j)} \Gamma(I_i, J_j, \{\gamma_p\}).$$

Also, we obtain the following Chinese operator:

$$\mathcal{U}_n^l(\gamma_1 \otimes \cdots \otimes \gamma_n) = \sum_m \sum_{\Gamma \in G_{n,m}^l} W_\Gamma \cdot \mathcal{U}_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n).$$

Theorem 5.1. *The $\mathcal{U} = (\mathcal{U}_n)$ gives the L_∞-morphism.*

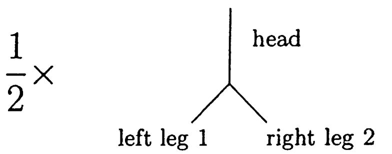
Proof. The same formalism of the proof of Formality theorem is available. □

Theorem 5.2. *The compatibility of cup products in the cohomology level holds.*

Proof. The same formalism of the proof in the case $T_{\mathrm{poly}}(\mathbb{R}^d)$, $D_{\mathrm{poly}}(\mathbb{R}^d)$ is available. □

5.3. Proof of BGRT conjecture

We consider the special Chinese graph 人 which we call *hito*.



The numbering of edges is as in the picture

$$L_\emptyset(\text{人}) = \{\text{head}\}, \qquad L_1(\text{人}) = \{\text{left leg}\}, \qquad L_2(\text{人}) = \{\text{right leg}\}.$$

Because of the IHX-relation, the 人 is a solution of Maurer–Cartan equation for the differential graded Lie algebra TCO. We put as follows:

$$\tilde{\text{人}} = \sum_{n \geq 0} \frac{1}{n!} \mathcal{U}_n^0(\text{人} \cdots \text{人}).$$

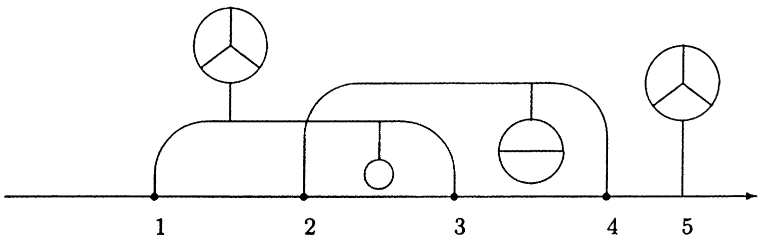
Then the $\tilde{\text{人}}$ gives the Maurer–Cartan solution for the differential graded Lie algebra DCO. It gives the deformation of the structure of associative algebras, which we denote by $(\mathcal{B}, \star)$.

The tangent map $T_{\text{人}}\mathcal{U} : (\mathcal{B}, m_{\mathcal{B}}) \to (\mathcal{B}, \star)$ is given. It preserves the product structure in the cohomology level by Theorem 5.2. Moreover, the differential of the $(\mathrm{TCO})_{\text{人}}$, which is given by the $\pm\mathrm{ad}(\curlywedge)$, vanishes as is checked easily. Hence, we conclude that the map $T_{\curlywedge}\mathcal{U} : (\mathcal{B}, m_{\mathcal{B}}) \to (\mathcal{B}, \star)$ preserve the product structure.

We can construct the algebra homomorphism of the $(\mathcal{A}, m_{\mathcal{A}})$ to the $(\mathcal{B}, \star)$ as follows.

We use the terminology 'Chinese character diagrams' for the diagrams which satisfies all conditions of Chinese character diagram but replaced circles by the directed lines (see [4, Section 3]). They are called 'linear diagrams' there.

We first construct the algebra homomorphism of the span(CCD) to the $(\mathcal{B}, \star)$. Take a Chinese character diagram Γ. Consider the trivalent vertices of Γ which is on the directed line of Γ. We give them the numbering which is increasing along the orientation of the arrow:



The group span(CCD) has the gradation span(CCD) $=$ span(CCD)^m by the number of the trivial vertices on the directed line.

We construct the Chinese operators $\tilde{\curlywedge}^{(m)}$ inductively. We put $\tilde{\curlywedge}^{(2)} = \tilde{\curlywedge}$. We put $\tilde{\curlywedge}^{(m)} = \tilde{\curlywedge}^{(m-1)} \circ_1 \tilde{\curlywedge}$, and $\tilde{\curlywedge}^{(0)}$, $\tilde{\curlywedge}^{(1)}$ are the empty graphs.

Any m-Chinese graph naturally gives the map of $\mathcal{A}$ to $\mathcal{B}$, i.e., removing the directed line, and either:

- combining legs L_i to the vertex i if $\#L_i = 1$,
- doing nothing if $L_i = \emptyset$,
- 0 if $\#L_i \geq 2$.

Thus the Chinese operator $\tilde{\curlywedge}^{(m)}$ gives the morphism I_{alg} from span(CCD)^m to $\mathcal{B}$.

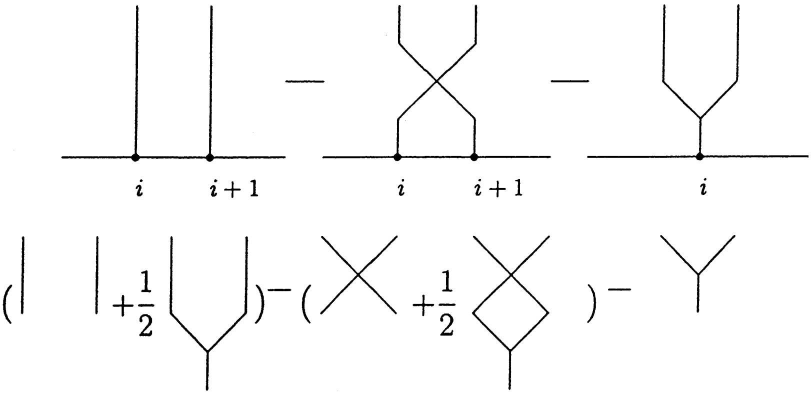
Proposition 5.3. *The morphism I_{alg} gives the algebra homomorphism from* span(CCD) *to* $(\mathcal{B}, \star)$.

Proof. It follows from the associativity of $\tilde{\curlywedge}$. Note that $(\tilde{\curlywedge} \circ_2 \tilde{\curlywedge}^{(l)})\tilde{\curlywedge}_1\tilde{\curlywedge}^{(m)} = \tilde{\curlywedge}^{(l+m)}$. □

Proposition 5.4. *The morphism I_{alg} gives the morphism from $\mathcal{A}$ to* $(\mathcal{B}, \star)$.

Proof. The associativity of $\tilde{\curlywedge}$ implies that $\tilde{\curlywedge}^{(m)} = \tilde{\curlywedge}^{(m-1)} \circ_i \tilde{\curlywedge}$.

We only have to prove that STU relations are preserved. The summands of $\tilde{\curlywedge}$ which contribute non-trivially are only empty-graph and $\curlywedge$. We can check directly the descent of the morphism by calculation, see the following picture: □



Hence we obtain the algebra morphism $(\mathcal{A}, m_{\mathcal{A}}) \to (\mathcal{B}, \star)$. On the other hand, we have the PBW morphism $\mathcal{B} \to \mathcal{A}$. Composing the morphisms, we obtain the algebra homomorphism of the $(\mathcal{B}, m_{\mathcal{A}})$ to $(\mathcal{B}, m_{\mathcal{B}})$.

Since the combinatorics of the construction is the same as that of Kontsevich [10], the obtained morphism is the Duflo–Kirillov morphism.

Hence we are done.

For further reading see [6–8,11].

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